

# GEOMETRY OF THIRD-ORDER ODES

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**ABSTRACT.** We address the problem of local geometry of third order ODEs modulo contact, point and fibre-preserving transformations of variables. Several new and already known geometries are described in a uniform manner by the Cartan method of equivalence. This includes conformal, Weyl and metric geometries in three and six dimensions and contact projective geometry in dimension three. Respective connections for these geometries are given and their curvatures are expressed by contact, point or fibre-preserving relative invariants of the ODEs.

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## 1. INTRODUCTION

This paper addresses the problem of geometry of third order ordinary differential equations (ODEs) which is stated as follows.

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**The geometry problem.** *Determine geometric structures defined by a class of equations*

$$y''' = F(x, y, y', y'')$$

*equivalent under certain type of transformations. Find relations between invariants of the ODEs and invariants of the geometric structures.*

One may consider equivalence with respect to several types of transformations, in this work we focus on three best known types: contact, point and fibre-preserving transformations. The fibre-preserving transformations are those which transform the independent variable  $x$  and the dependent variable  $y$  in such a way that the notion of the independent variable is retained, that is the transformation of  $x$  is a function of  $x$  only:

$$(1) \quad x \mapsto \bar{x} = \chi(x), \quad y \mapsto \bar{y} = \phi(x, y).$$

The transformation rules for the derivatives are already uniquely defined by above formulae. Let us define the total derivative to be

$$\mathfrak{D} = \partial_x + y' \partial_y + y'' \partial_{y'} + y''' \partial_{y''}.$$

Then

$$(2a) \quad y' \mapsto \frac{d\bar{y}}{d\bar{x}} = \frac{\mathfrak{D}\phi}{\mathfrak{D}\chi},$$

$$(2b) \quad y'' \mapsto \frac{d^2\bar{y}}{d\bar{x}^2} = \frac{\mathfrak{D}}{\mathfrak{D}\chi} \left( \frac{\mathfrak{D}\phi}{\mathfrak{D}\chi} \right),$$

$$(2c) \quad y''' \mapsto \frac{d^3\bar{y}}{d\bar{x}^3} = \frac{\mathfrak{D}}{\mathfrak{D}\chi} \left( \frac{\mathfrak{D}}{\mathfrak{D}\chi} \left( \frac{\mathfrak{D}\phi}{\mathfrak{D}\chi} \right) \right).$$

The point transformations of variables mix  $x$  and  $y$  in an arbitrary way

$$(3) \quad x \mapsto \bar{x} = \chi(x, y), \quad y \mapsto \bar{y} = \phi(x, y),$$

with the derivatives transforming as in (2). The contact transformations are more general yet. Not only they augment the independent and the dependent variables but also the first derivative

$$\begin{aligned} x &\mapsto \bar{x} = \chi(x, y, y'), \\ y &\mapsto \bar{y} = \phi(x, y, y'), \\ y' &\mapsto \frac{d\bar{y}}{d\bar{x}} = \psi(x, y, y'). \end{aligned}$$

However, the functions  $\chi$ ,  $\phi$  and  $\psi$  are not arbitrary here but subjected to (2a) which now yields two additional constraints

$$\psi = \frac{\mathfrak{D}\phi}{\mathfrak{D}\chi} \iff \begin{aligned} \psi \chi_{y'} &= \phi_{y'}, \\ \psi(\chi_x + y' \chi_y) &= \phi_x + y' \phi_y, \end{aligned}$$

guaranteeing that  $d\bar{y}/d\bar{x}$  really transforms like first derivative. With these conditions fulfilled second and third derivative transform through (2b) – (2c). We always assume in this work that ODEs are defined locally by a smooth real function  $F$  and are considered apart from singularities. The transformations are always assumed to be local diffeomorphisms.

A pioneering work on geometry of ODEs of arbitrary order is Karl Wünschmann's PhD thesis [1] written under supervision of F. Engel in 1905. In this paper K. Wünschmann observed that solutions of an  $n$ th-order ODE  $y^{(n)} = F(x, y, y', \dots, y^{(n-1)})$  may be considered as both curves  $y = y(x, c_0, c_1, \dots, c_{n-1})$  in the  $xy$  space and points  $c = (c_0, \dots, c_{n-1})$  in the solution space  $\mathbb{R}^n$  parameterized by values of the constants of integration  $c_i$ . He defined a relation of  $k$ th-order contact between infinitesimally close solutions considered as curves; two solutions  $y(x)$

and  $y(x) + dy(x)$  corresponding to  $c$  and  $c + dc$  have the  $k$ th-order contact if their  $k$ th jets coincide at some point  $(x_0, y_0)$ . Wünschmann's main question was how the property of having  $(n - 2)$ nd contact for  $n = 3, 4$  and  $5$  might be described in terms of the solution space. In particular he examined third-order ODEs and showed that there is a distinguished class of ODEs satisfying certain condition for the function  $F$ , which we call the Wünschmann condition. For a third-order ODE in this class, the condition of having first order contact is described by a second order Monge equation for  $dc$ . This Monge equation is nothing but the condition that the vector defined by two infinitesimally close points  $c$  and  $c + dc$  is null with respect to a Lorentzian conformal metric on the solution space. The last observation, although not contained in Wünschmann's work, follows immediately from his reasoning and was later made by S.-S. Chern [5], who cited Wünschmann's thesis.

The main contribution to the issue of point and contact geometry of third-order ODEs was made by E. Cartan and S.-S. Chern in their classical papers [2, 3, 4] and [5]. E. Cartan [3] considered a third-order ODE modulo point transformations and, applying his method of equivalence, constructed a 7-dimensional manifold  $P$  together with a fixed coframe  $\theta^1, \theta^2, \theta^3, \theta^4, \Omega_1, \Omega_2, \Omega_3$  which encodes all the point invariant information about the ODE. This means that two ODEs are point equivalent if and only if their associated coframes are diffeomorphic. In the language of contemporary differential geometry  $P$  is a principal bundle  $H_3 \rightarrow P \rightarrow J^3$  over the second jet space with the structure group  $H_3 = \mathbb{R} \times (\mathbb{R} \ltimes \mathbb{R})$  while the coframe defines a  $\mathfrak{so}(2, 1) \oplus \mathbb{R}^3$ -valued<sup>1</sup> Cartan connection on  $P$ . The point invariant information about the ODE is contained in the curvature of this connection and its coframe derivatives. E. Cartan observed that some equations have a non-trivial geometry on their solution spaces, namely a 3-dimensional Einstein-Weyl geometry in Lorentzian signature. In order to possess it an ODE must satisfy two point invariant differential conditions on the function  $F$ , one of them being the Wünschmann condition. E. Cartan also showed that the Weyl connection for this geometry may be immediately obtained from the coframe  $\theta^1, \dots, \Omega_3$ . Following Cartan's reasoning S.-S. Chern studied third-order ODEs modulo contact transformations and constructed a 10-dimensional bundle  $P \rightarrow J^2$  equipped with a coframe  $\theta^1, \dots, \theta^4, \Omega_1, \dots, \Omega_6$ . Iff an ODE satisfies the Wünschmann condition then it has a three-dimensional Lorentzian conformal geometry on the solution space while the coframe becomes the  $\mathfrak{o}(3, 2)$ -valued normal conformal connection for the geometry. In both these cases the conformal metric is precisely the metric appearing implicitly in K. Wünschmann's thesis. Later H. Sato and A. Yoshikawa [43] applying N. Tanaka's theory [46] constructed a Cartan normal connection for arbitrary third-order ODEs (not only of the Wünschmann type) and showed how its curvature is expressed by the contact relative invariants.

The Lorentzian geometry on the solution space was rediscovered fifty years after S.-S. Chern from the perspective of General Relativity. In a series of papers [18, 19, 20, 24] E.T. Newman et al developed the Null Surface Formulation (NSF), an alternate approach to General Relativity. In NSF the basic concept is a family of hypersurfaces on a manifold  $M^4$ ; the hypersurfaces are defined as level sets  $Z(x^\mu, s, s^*) = \text{const}$  of a real function<sup>2</sup>  $Z$  on  $M \times S^2$ , where  $(x^\mu) \in M^4$ ,  $s \in S^2$ . Starting from these data a Lorentzian *conformal* metric is constructed by the property that these hypersurfaces are its null hypersurfaces. The function

<sup>1</sup>Notation is explained on pages 7 – 8.

<sup>2</sup>Here  $s^*$  is the complex conjugate of  $s$ .

$Z(x^\mu, s, s^*)$  is interpreted as the general solution to a system of PDEs

$$\begin{aligned} Z_{ss} &= \Lambda(s, s^*, Z, Z_s, Z_{s^*}, Z_{ss^*}), \\ Z_{s^*s^*} &= \Lambda^*(s, s^*, Z, Z_s, Z_{s^*}, Z_{ss^*}) \end{aligned}$$

for a real function  $Z(s)$  of a complex variable  $s$ . Consequently, coordinates  $x^\mu$  are treated as constants of integration, which turns  $M^4$  into the solution space of the PDEs. One of main results of NSF was proving that a family of hypersurfaces is a family of null hypersurfaces for a Lorentzian conformal metric on  $M^4$  if and only if  $\Lambda$  satisfies two differential conditions, so called metricity conditions, which may be viewed as a generalization of the Wünschmann condition. It is remarkable that three-dimensional version of the NSF leads immediately to third-order ODEs, [47, 21, 22, 23]. In this case there is a one-parameter family of surfaces in  $M^3$  given by  $Z(x^i, s) = \text{const}$ , where  $(x^i) \in M^3$  and  $s \in S^1$  are real.  $Z(x^i, s)$  is identified with the general solution of a third order ODE  $Z_{sss} = \Lambda(s, Z, Z_s, Z_{ss})$ ,  $M^3$  is the solution space and  $M^3 \times S^1$  is identified with  $J^2$ . The construction of a conformal geometry from the family of null surfaces is fully equivalent to Chern's construction and one metricity condition obtained for  $\Lambda$  in this case is precisely the Wünschmann condition.

Apart from the 3-dimensional Lorentzian conformal geometry, there has been some interest in 3-dimensional Einstein-Weyl geometry, mainly from the perspective of the theory of twistors and integrable systems by N. Hitchin [29], R. Ward [49], C. LeBrun [33] and P. Tod, M. Dunajski et al [31, 14, 15]; for discussion of the link between the Einstein-Weyl spaces and third-order ODEs see [38] and [47].

P. Nurowski, following the ideas of E. Cartan, proposed a programme of systematic study of geometries related to differential equations, including second- and third-order ODEs. In this programme, [23, 27, 28, 38, 40], both new and already known geometries associated with differential equations are supposed to be constructed by the Cartan equivalence method and are to be characterized in the language of Cartan connections associated with them. In particular in [38] new examples of geometries associated with ordinary differential equations were given, including a conformal geometry with special holonomy  $G_2$  from ODEs of the Monge type. Partial results on geometries of third-order ODEs were given in [38, 23, 27] but the full analysis of these geometries has not been published so far and this paper aims to fill this gap.

Geometry of third-order ODEs is a part of broader issue of geometry of differential equations in general. Regarding ODEs of order two, we owe classical results including construction of point invariants to S. Lie [34] and M. Tresse [48]. In particular E. Cartan [10] constructed a two-dimensional projective differential geometry on the solution spaces of some second-order ODEs. This geometry was further studied in [37] and [40], the latter paper pursues the analogy between geometry of three-dimensional CR structures and second-order ODEs and provides a construction of counterparts of the Fefferman metrics for the ODEs. Classification of second-order ODEs possessing Lie groups of fibre-preserving symmetries was done by L. Hsu and N. Kamran [30]. Geometry on the solution space of certain four-order ODEs (satisfying two differential conditions), which is given by the four-dimensional irreducible representation of  $GL(2, \mathbb{R})$  and has exotic  $GL(2, \mathbb{R})$  holonomy was discovered and studied by R. Bryant [6], see also [39]. The  $GL(2, \mathbb{R})$  geometry of fifth-order ODEs has been recently studied by M. Godliński and P. Nurowski [28].

The more general problems yet are existence and properties of geometry on solution spaces of arbitrary ODEs. The problem of existence was solved by B.

Doubrov [12], who proved that an  $n$ th-order ODE  $n \geq 3$ , modulo contact transformations, has a geometry based on the irreducible  $n$ -dimensional representation of  $GL(2, \mathbb{R})$  provided that it satisfies  $n - 2$  scalar differential conditions. An implicit method of constructing these conditions was given in [11]. Properties of the  $GL(2, \mathbb{R})$  geometries of ODEs are still an open problem; they were studied in [28], where the Doubrov conditions were interpreted as higher order counterparts of the Wünschmann condition, and by M. Dunajski and P. Tod [16].

Almost all the above papers deal with geometries on solution spaces but one can also consider other geometries, including those defined on various jet spaces. The most general result on such geometries [13] comes from T. Morimoto's nilpotent geometry [35, 36]. It concludes that with a system of ODEs there is associated a filtration on a suitable jet space together with a canonical Cartan connection.

**1.1. Results of the paper.** We start from the equivalence problems formulated in terms of  $G$ -structures (12) – (14) and apply Cartan's method to obtain manifolds  $P$  equipped with the coframes encoding all the invariant information about ODEs. Next we show how to read the principal bundle structures of these manifolds over distinct bases. Usually it is the structure over the solution space  $\mathcal{S}$  which is the most interesting, but we also consider structures over  $J^1$ ,  $J^2$  and certain six-dimensional manifold  $M^6$ , which appears naturally. When the structure of a bundle is established then the invariant coframe defines a Cartan connection on  $P \rightarrow \text{base}$ , usually under additional conditions playing similar role to the Wünschmann condition. In order to obtain the geometries and the Wünschmann-like conditions we often apply the method of construction by Lie transport and projection. The most symmetric cases when the connections are flat or have constant curvature provide homogeneous models for the geometries. Since the dimension of  $\mathcal{S}$  and  $J^1$  is three and we build geometries with at least two-dimensional structural group then the homogeneous models are given by the ODEs with at least five-dimensional symmetry group.

*Contact geometries.* Section 3 is devoted to the geometries of the ODEs modulo contact transformations. The only equations possessing at least five-dimensional contact symmetry group are the linear equations with constant coefficients, that is

$$y''' = 0$$

with the symmetry group  $O(3, 2)$  and

$$y''' = -2\mu y' + y, \quad \mu \in \mathbb{R},$$

mutually non-equivalent for distinct  $\mu$ , with the symmetry group  $\mathbb{R}^2 \ltimes_{\mu} \mathbb{R}^3$ . Sections 3.1 to 3.6 discuss geometries whose homogeneous model is generated by  $y''' = 0$ . In section 3.1 we state the main theorem in that section, theorem 3.1, which describes the geometry on  $J^2$ . It may be recapitulated as follows.

**Theorem** (Theorem 3.1). *The contact invariant information about an equation  $y''' = F(x, y, y', y'')$  is given by the following data*

- i) *The principal fibre bundle  $H_6 \rightarrow P \rightarrow J^2$ , where  $\dim P = 10$  and  $H_6$  is a six-dimensional subgroup of  $O(3, 2)$*
- ii) *The coframe  $(\theta^1, \theta^2, \theta^3, \theta^4, \Omega_1, \Omega_2, \Omega_3, \Omega_4, \Omega_5, \Omega_6)$  on  $P$  which defines the  $\mathfrak{o}(3, 2) \cong \mathfrak{sp}(4, \mathbb{R})$  Cartan normal connection  $\hat{\omega}^c$  on  $P$ .*

*The coframe and the connection  $\hat{\omega}_c$  are given explicitly in terms of  $F$  and its derivatives. There are two basic relative invariants for this geometry: the Wünschmann invariant and  $F_{y''y''y''y''}$ .*

This theorem is almost identical to the result proved in [43] and the only new element we add here is the explicit formula for the connection. For completeness,

section 3.2 contains a proof of the theorem, which follows S.-S. Chern's construction of the coframe and the construction of the normal connection of [43].

Sections 3.4 to 3.6 discuss next three geometries generated by  $\widehat{\omega}^c$ , those on  $J^1$ ,  $\mathcal{S}$  and certain six-dimensional manifold  $M^6$  which appears in a natural way. These results are summarized as follows.

**Theorem.** *The connection  $\widehat{\omega}^c$  of theorem 3.1 has fourfold interpretation.*

1. *It is always the normal  $\mathfrak{o}(3,2)$  Cartan connection on  $J^2$  (Chern-Sato-Yoshikawa construction.)*
2. *If the ODE has vanishing Wünschmann condition then  $\widehat{\omega}^c$  is the normal Lorentzian conformal connection for the Lorentzian structure on the solution space (Chern-NSF construction.)*
3. *If the ODE satisfies  $F_{y''y''y''y''} = 0$  then  $\widehat{\omega}^c$  becomes the normal Cartan connection for the contact projective structure on  $J^1$  generated by the family of solutions of the ODE.*
4.  *$\widehat{\omega}^c$  is the  $\mathfrak{sp}(4, \mathbb{R})$ -part of the  $\mathfrak{o}(4,4)$  normal conformal connection for a six-dimensional split conformal geometry on  $M^6$  with special holonomy  $\mathfrak{sp}(4, \mathbb{R}) \oplus \mathbb{R}^5$ .*

In section 3.7 we turn to geometries, whose homogeneous models are provided by the equation  $y''' = -2\mu y' + y$ . Following Chern we reduce the bundle  $P$  to its five-dimensional subbundle. Then we find that

**Theorem** (Theorem 3.15). *Every ODE satisfying some contact invariant condition  $\mathbf{a}^c[F] = \mu = \text{const}$  has a  $\mathbb{R}^2$  geometry on its solution space together with a  $\mathbb{R}^2$  linear connection from the invariant coframe. The action of the algebra  $\mathbb{R}^2$  on  $\mathcal{S}$  is given by*

$$\begin{pmatrix} u & v & 0 \\ -\mu v & u & v \\ v & -\mu v & u \end{pmatrix}.$$

This geometry seems to be a generalization of Chern's 'cone geometry' which was associated with the equation  $y''' = -y$  and briefly mentioned to exist for arbitrary ODEs. In our construction the action of  $\mathbb{R}^2$  depends on the characteristic polynomial of respective linear equation, and we get a real cone geometry provided that it has three distinct roots.

*Point geometries.* In section 4 we study the geometries associated with the ODEs modulo point transformations. Sections 4.1 to 4.4 deal with geometries modelled on  $y''' = 0$ . Our approach is analogous to the contact case and results are similar. They may be summarized as follows.

**Theorem.** *The following statements hold*

1. *The point invariant information about  $y''' = F(x, y, y', y'')$  is given by the seven-dimensional principal bundle  $H_3 \rightarrow P \rightarrow J^2$  together with the coframe  $\theta^1, \theta^2, \theta^3, \theta^4, \Omega_1, \Omega_2, \Omega_3$  on  $P$ , which defines the  $\mathfrak{co}(2,1) \oplus \mathbb{R}^3$  Cartan connection  $\widehat{\omega}^p$  (Cartan construction.)*
2. *If the ODE has vanishing Wünschmann and Cartan invariants then it has the Einstein-Weyl geometry on  $\mathcal{S}$  and the Weyl connection is given by  $\widehat{\omega}^p$  (Cartan construction.)*
3. *If the ODE satisfies  $F_{y''y''y''y''} = 0$  then it has the point-projective structure on  $J^1$  generated by the family of solutions of the ODE.*
4. *For any ODE there exists the split signature six-dimensional Weyl geometry on certain manifold  $M^6$ , which is never Einstein.*

A new construction, which does not have a contact counterpart, is considered in section 4.5. This is a Lorentzian *metric structure* on the solution space  $\mathcal{S}$ . Its construction follows immediately from the Einstein-Weyl geometry. If the Ricci scalar of the Weyl connection is non-zero, then it is a weighted conformal function and may be fixed to a constant by an appropriate choice of the conformal gauge. The homogeneous models of this geometry are associated with

$$y''' = \frac{3y''^2}{2y'}$$

if the Ricci scalar is negative, and

$$y''' = \frac{3y''^2 y'}{y'^2 + 1}$$

if the Ricci scalar is positive. Their point symmetry groups are  $O(2, 2)$  and  $O(4)$  respectively. Both these equations are contact equivalent to  $y''' = 0$ .

*Fibre-preserving geometries.* Section 5 is devoted to the geometries of ODEs modulo fibre-preserving transformations. We obtain a seven-dimensional bundle and the  $\mathfrak{co}(2, 1) \oplus \mathbb{R}^3$ -valued Cartan connection  $\hat{\omega}^f$  on it. Since both  $\hat{\omega}^f$  and  $\hat{\omega}^p$  of the point case take value in the same algebra these cases are very similar to each other. Indeed, we show that one can recover  $\hat{\omega}^f$  from  $\hat{\omega}^p$  just by appending one function on the bundle. As a consequence, the geometries of the fibre-preserving case are obtained from their point counterparts by appending the object generated by the function.

We did not study obvious or not interesting geometries. In the point and fibre-preserving case geometries of  $y''' = -2\mu y' + y$  are the same to what we have in the contact case, since the respective symmetry groups are the same. Also the fibre-preserving geometry on  $J^1$  does not seem to be worth studying.

To summarize, the following material contained in this work is new: sections 3.5 to 3.7 excluding theorem 3.13 and sections 4.3 to 5.2. Other sections contain a reformulation and an extension of already known results.

All our calculations were performed or checked using the symbolic calculations program Maple.

**1.2. Notation.** In what follows we use the following symbols, in particular  $W$  denotes the Wünschmann invariant.

$$\begin{aligned} (4) \quad & F = F(x, y, p, q), \\ & \mathcal{D} = \partial_x + p\partial_y + q\partial_p + F\partial_q, \\ (5) \quad & K = \frac{1}{6}\mathcal{D}F_q - \frac{1}{9}F_q^2 - \frac{1}{2}F_p, \\ (6) \quad & L = \frac{1}{3}F_{qq}K - \frac{1}{3}F_qK_q - K_p - \frac{1}{3}F_{qy}, \\ (7) \quad & M = 2K_{qq}K - 2K_{qy} + \frac{1}{3}F_{qq}L - \frac{2}{3}F_qL_q - 2L_p, \\ (8) \quad & W = \left(\mathcal{D} - \frac{2}{3}F_q\right)K + F_y, \\ (9) \quad & Z = \frac{\mathcal{D}W}{W} - F_q. \end{aligned}$$

Parentheses denote sets of objects:  $(a_1, \dots, a_k)$  is the set consisting of  $a_1, \dots, a_k$ . In particular this symbol denotes bases of vector spaces as well as coordinate systems, frames and coframes on manifolds. The linear span of vectors or covectors  $a_1, \dots, a_k$  is denoted by  $\langle a_1, \dots, a_k \rangle$ . If  $a_1, \dots, a_k$  are vector fields or one-forms on a manifold, then the above symbol denotes the distribution or the simple ideal generated by them. The symmetric tensor product of two one-forms or vector fields  $\alpha$  and  $\beta$  is denoted by  $\alpha\beta = \frac{1}{2}(\alpha \otimes \beta + \beta \otimes \alpha)$ . The symbols  $A_{(\mu\nu)}$  and  $A_{[\mu\nu]}$  denote

symmetrization and antisymmetrization of a tensor  $A_{\mu\nu}$  respectively. For a metric  $g$  of signature  $(k, l)$  the group  $CO(k, l)$  is defined to be

$$CO(k, l) = \{A \in GL(k + l, \mathbb{R}) \mid A^T g A = e^\lambda g, \quad \lambda \in \mathbb{R}\}.$$

Its Lie algebra

$$\mathfrak{co}(k, l) = \{a \in \mathfrak{gl}(k + l, \mathbb{R}) \mid a^T g + ga = \lambda g, \quad \lambda \in \mathbb{R}\}.$$

A semidirect product of two Lie groups  $G$  and  $H$ , where  $G$  acts on  $H$  is denoted by  $G \ltimes H$ . A semidirect product of their Lie algebras is denoted by  $\mathfrak{g} \oplus \mathfrak{h}$ . If the action depends of a parameter  $\mu$  then we add an subscript:  $\ltimes_\mu$  and  $\oplus_\mu$ .

## 2. TOWARDS CARTAN CONNECTIONS

**2.1. ODEs as G-structures on  $J^2$ .** Following Cartan and Chern we begin with the space  $J^2$  of second jets of curves in  $\mathbb{R}^2$  (see [41] for description of jet spaces) with coordinate system  $(x, y, p, q)$  where  $p$  and  $q$  denote first and second derivative  $y'$  and  $y''$  for a curve  $x \mapsto (x, y(x))$  in  $\mathbb{R}^2$ , so that this curve lifts to a curve  $x \mapsto (x, y(x), y'(x), y''(x))$  in  $J^2$ . Any solution  $y = f(x)$  of  $y''' = F(x, y, y', y'')$  is uniquely defined by a choice of  $f(x_0)$ ,  $f'(x_0)$  and  $f''(x_0)$  at some  $x_0$ . Since that choice is equivalent to a choice of a point in  $J^2$ , there passes exactly one solution  $(x, f(x), f'(x), f''(x))$  through any point of  $J^2$ . Therefore the solutions form a (local) congruence on  $J^2$ , which can be described by its annihilating simple ideal. Let us choose a coframe  $(\omega^i)$  on  $J^2$ :

$$\begin{aligned} \omega^1 &= dy - p dx, \\ \omega^2 &= dp - q dx, \\ \omega^3 &= dq - F(x, y, p, q) dx, \\ \omega^4 &= dx. \end{aligned} \tag{10}$$

Each solution  $y = f(x)$  is fully described by the two conditions: forms  $\omega^1, \omega^2, \omega^3$  vanish on the curve  $t \mapsto (t, f(t), f'(t), f''(t))$  and, since this defines a solution modulo transformations of  $x$ ,  $\omega^4 = dt$  on this curve.

Suppose now that a equation  $y''' = F(x, y, y', y'')$  undergoes a contact, point or fibre-preserving transformation. Then (10) transform by

$$\begin{aligned} \omega^1 &\mapsto \bar{\omega}^1 = u_1 \omega^1, \\ \omega^2 &\mapsto \bar{\omega}^2 = u_2 \omega^1 + u_3 \omega^2, \\ \omega^3 &\mapsto \bar{\omega}^3 = u_4 \omega^1 + u_5 \omega^2 + u_6 \omega^3, \\ \omega^4 &\mapsto \bar{\omega}^4 = u_8 \omega^1 + u_9 \omega^2 + u_7 \omega^4, \end{aligned} \tag{11}$$

with some functions  $u_1, \dots, u_9$  defined on  $J^2$  and determined by a particular choice of transformation, for instance

$$\begin{aligned} u_1 &= \phi_y - \psi \chi_y, \\ u_7 &= \mathfrak{D}\chi, \quad u_8 = \chi_y, \quad u_9 = \chi_p. \end{aligned}$$

In particular,  $u_9 = 0$  in the point case and  $u_8 = u_9 = 0$  in the fibre-preserving case. Since the transformations are non-degenerate, the condition  $u_1 u_3 u_5 u_7 \neq 0$  is always satisfied and transformations (11) form local pseudogroups. Thus we have.

**Lemma 2.1.** *A class of contact equivalent third-order ODEs is a local G-structure<sup>3</sup>  $G_c \times J^2$  defined by the property that the coframe  $(\omega^1, \omega^2, \omega^3, \omega^4)$  belongs to it and*

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<sup>3</sup>Here we use local trivializations of G-structures.



the structure group is given by

$$(12) \quad G_c = \begin{pmatrix} u_1 & 0 & 0 & 0 \\ u_2 & u_3 & 0 & 0 \\ u_4 & u_5 & u_6 & 0 \\ u_8 & u_9 & 0 & u_7 \end{pmatrix}.$$

For a class of point equivalent ODEs the structure group  $G_p \subset G_c$  is given by

$$(13) \quad G_p = \begin{pmatrix} u_1 & 0 & 0 & 0 \\ u_2 & u_3 & 0 & 0 \\ u_4 & u_5 & u_6 & 0 \\ u_8 & 0 & 0 & u_7 \end{pmatrix},$$

whereas for a class of fibre-preserving equivalent ODEs the structure group  $G_f \subset G_p$  is given by

$$(14) \quad G_f = \begin{pmatrix} u_1 & 0 & 0 & 0 \\ u_2 & u_3 & 0 & 0 \\ u_4 & u_5 & u_6 & 0 \\ 0 & 0 & 0 & u_7 \end{pmatrix}.$$

**2.2. Cartan connections.** The version of Cartan's equivalence method [9] we employ below is explained in books by R. Gardner [25] and P. Olver [42]. Some its aspects are also discussed by S. Sternberg [44] and S. Kobayashi [32]. Its idea is the following: starting from  $G_c \times J^2$  (or  $G_p \times J^2$  or  $G_f \times J^2$ ) one constructs a new principal bundle  $H \rightarrow P \rightarrow J^2$  (with a new group  $H$ ) equipped with one fixed coframe  $(\theta^i, \Omega_\mu)$  built in some geometric way and such that it encodes all the local invariant information about  $G_c \times J^2$  (or  $G_p \times J^2$  or  $G_f \times J^2$  respectively) through its structural equations. This leads to the notion of Cartan connection, defined here after [32].

**Definition 2.2.** Let  $H \rightarrow P \rightarrow M$  be a principal bundle and let  $G$  be a Lie group such that  $H$  is its closed subgroup and  $\dim G = \dim P$ . A Cartan connection of type  $(G, H)$  on  $P$  is a one-form  $\hat{\omega}$  taking values in the Lie algebra  $\mathfrak{g}$  of  $G$  and satisfying the following conditions:

- i)  $\hat{\omega}_u : T_u P \rightarrow \mathfrak{g}$  for every  $u \in P$  is an isomorphism of vector spaces
- ii)  $A^* \lrcorner \hat{\omega} = A$  for every  $A \in \mathfrak{h}$  and the corresponding fundamental field  $A^*$
- iii)  $R_h^* \hat{\omega} = \text{Ad}(h^{-1}) \hat{\omega}$  for  $h \in H$ .

The curvature of a Cartan connection is defined as follows

$$\hat{K}(X, Y) = d\hat{\omega}(X, Y) + \frac{1}{2}[\hat{\omega}(X), \hat{\omega}(Y)].$$

The curvature is horizontal which means that it vanishes on each fundamental vector field  $A^*$ :

$$A^* \lrcorner \hat{K} = 0.$$

Horizontality of the curvature is locally equivalent with property iii) of the connection. Cartan connections with vanishing curvature are called flat.

**Example 2.3.** A standard example of Cartan connection is obtained by taking the principal fibre bundle of a Lie group  $P = G$  over its homogeneous space  $M = G/H$  and by defining  $\hat{\omega}$  to be the Maurer-Cartan form  $\hat{\omega} = g^{-1}dg$ ,  $g \in G$ . So defined  $\hat{\omega}$  is a flat Cartan connection on  $P$ .

In general, the curvature does not have to vanish and a Cartan connection is an object that generalizes the notion of the Maurer-Cartan form on a Lie group.

Two 3rd order ODEs  $y''' = F(x, y, y', y'')$  and  $y''' = \bar{F}(x, y, y', y'')$  are contact/fibre-preserving equivalent if and only if their associated Cartan

connections are diffeomorphic, that is there exists a local bundle diffeomorphism  $\Phi: \bar{P} \supset \bar{\mathcal{O}} \rightarrow \mathcal{O} \subset P$  such that  $\Phi^*\hat{\omega} = \bar{\omega}$ . Technically speaking curvature coefficients are rational functions of vertical bundle coordinates  $(u_\mu)$  and relative invariants defined as follows.

**Definition 2.4.** A contact/point/fibre-preserving relative invariant  $I[F]$  of  $y''' = F(x, y, y', y'')$  is a function of  $F$  and its derivatives such that if  $I[F] \equiv 0$  for an equation  $y''' = F(x, y, y', y'')$  then  $I[\bar{F}] \equiv 0$  for every equation  $y''' = \bar{F}(x, y, y', y'')$  contact/point/fibre-preserving equivalent to it.

From relative invariants contained in the curvature of the associated Cartan connection one may recover the full set of invariants by taking consecutive coframe derivatives.

In theorem 3.1 and proposition 3.10 we need a concept of normal Cartan connection in the sense of N. Tanaka. We briefly remind this notion below. For details of Tanaka's theory see [45, 46].

**Definition 2.5.** A semisimple Lie algebra  $\mathfrak{g}$  is graded if it has a vector space decomposition

$$\mathfrak{g} = \mathfrak{g}_{-k} \oplus \dots \oplus \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \dots \oplus \mathfrak{g}_k$$

such that

$$[\mathfrak{g}_i, \mathfrak{g}_j] \subset \mathfrak{g}_{i+j}$$

and  $\mathfrak{g}_{-k} \oplus \dots \oplus \mathfrak{g}_{-1}$  is generated by  $\mathfrak{g}_{-1}$ .

Let us suppose that  $\mathfrak{g}$  is a semisimple graded Lie algebra and denote  $\mathfrak{m} = \mathfrak{g}_{-k} \oplus \dots \oplus \mathfrak{g}_{-1}$ ,  $\mathfrak{h} = \mathfrak{g}_0 \oplus \dots \oplus \mathfrak{g}_k$ . Let us consider a  $\mathfrak{g}$  valued Cartan connection  $\hat{\omega}$  on a bundle  $H \rightarrow P \rightarrow M$ , where the Lie algebra of  $H$  is  $\mathfrak{h}$ . Fix a point  $p \in P$ . The decomposition  $\mathfrak{g} = \mathfrak{m} \oplus \mathfrak{h}$  defines in  $T_p P$  the complement  $\mathcal{H}_p$  of the vertical space  $\mathcal{V}_p$ . Therefore we have  $T_p P = \mathcal{V}_p \oplus \mathcal{H}_p$ ,  $\hat{\omega}(\mathcal{V}_p) = \mathfrak{h}$  and  $\hat{\omega}(\mathcal{H}_p) = \mathfrak{m}$ . The curvature  $\hat{K}_p = (d\hat{\omega} + \hat{\omega} \wedge \hat{\omega})_p$  at  $p$  is then characterized by the tensor  $\kappa_p \in \text{Hom}(\wedge^2 \mathfrak{m}, \mathfrak{g})$  given by

$$(15) \quad \kappa_p(A, B) = \hat{K}_p(\hat{\omega}_p^{-1}(A), \hat{\omega}_p^{-1}(B)), \quad A, B \in \mathfrak{m}.$$

The function  $\kappa: P \rightarrow \text{Hom}(\wedge^2 \mathfrak{m}, \mathfrak{g})$  is called the structure function.

In the space  $\text{Hom}(\wedge^2 \mathfrak{m}, \mathfrak{g})$  let us define  $\text{Hom}^1(\wedge^2 \mathfrak{m}, \mathfrak{g})$  to be the space of all  $\alpha \in \text{Hom}(\wedge^2 \mathfrak{m}, \mathfrak{g})$  fulfilling

$$\alpha(\mathfrak{g}_i, \mathfrak{g}_j) \subset \mathfrak{g}_{i+j+1} \oplus \dots \oplus \mathfrak{g}_k \quad \text{for } i, j < 0.$$

Since the Killing form  $B$  of  $\mathfrak{g}$  is non-degenerate and satisfies  $B(\mathfrak{g}_p, \mathfrak{g}_q) = 0$  for  $p \neq -q$ , one can identify  $\mathfrak{m}^*$  with  $\mathfrak{g}_1 \oplus \dots \oplus \mathfrak{g}_k$ . For a basis  $(e_1, \dots, e_m)$  of  $\mathfrak{m}$  let  $(e_1^*, \dots, e_m^*)$  denote the unique basis of  $\mathfrak{g}_1 \oplus \dots \oplus \mathfrak{g}_k$  such that  $B(e_i, e_j^*) = \delta_{ij}$ . Tanaka considered the following complex

$$\dots \longrightarrow \text{Hom}(\wedge^{q+1} \mathfrak{m}, \mathfrak{g}) \xrightarrow{\partial^*} \text{Hom}(\wedge^q \mathfrak{m}, \mathfrak{g}) \longrightarrow \dots$$

with  $\partial^*: \text{Hom}(\wedge^{q+1} \mathfrak{m}, \mathfrak{g}) \rightarrow \text{Hom}(\wedge^q \mathfrak{m}, \mathfrak{g})$  given by the following formula

$$\begin{aligned} (\partial^* \alpha)(A_1 \wedge \dots \wedge A_q) &= \sum_i [e_i^*, \alpha(e_i \wedge A_1 \wedge \dots \wedge A_q)] \\ &+ \frac{1}{2} \sum_{i,j} \alpha([e_j^*, A_i]_{\mathfrak{m}} \wedge e_j \wedge A_1 \wedge \dots \wedge \hat{A}_i \wedge \dots \wedge A_q), \end{aligned}$$

where  $\alpha \in \text{Hom}(\wedge^q \mathfrak{m}, \mathfrak{g})$ ,  $A_1, \dots, A_q \in \mathfrak{m}$ ,  $(e_i)$  is any basis in  $\mathfrak{m}$  and  $[\cdot, \cdot]_{\mathfrak{m}}$  denotes the  $\mathfrak{m}$ -component of the bracket with respect to the decomposition  $\mathfrak{g} = \mathfrak{m} \oplus \mathfrak{h}$ . Finally, N. Tanaka [46] introduced the notion of normal connection, the definition below is given in the language of [7].

**Definition 2.6.** A Cartan connection  $\widehat{\omega}$  as above is normal if its structure function  $\kappa$  fulfills the following conditions

- i)  $\kappa \in \text{Hom}^1(\wedge^2 \mathfrak{m}, \mathfrak{g}),$
- ii)  $\partial^* \kappa = 0.$

### 3. GEOMETRIES OF ODES MODULO CONTACT TRANSFORMATIONS OF VARIABLES

**3.1. Cartan connection on ten-dimensional bundle.** We formulate a theorem about the main structure which is associated with third-order ODEs modulo contact transformations of variables, an  $\mathfrak{sp}(4, \mathbb{R})$  Cartan connection on the bundle  $P^c \rightarrow J^2$ . This structure will serve as a starting point for further analyzing of geometries of ODEs.

**Theorem 3.1.** *To every third order ODE  $y''' = F(x, y, y', y'')$  there are associated the following data.*

- i) *The principal fibre bundle  $H_6 \rightarrow P^c \rightarrow J^2$ , where  $\dim P^c = 10$  and  $H_6$  is the following six-dimensional subgroup of  $SP(4, \mathbb{R})$*

$$(16) \quad H_6 = \begin{pmatrix} \sqrt{u_1}, & \frac{1}{2} \frac{u_2}{\sqrt{u_1}}, & -\frac{1}{2} \frac{u_4}{\sqrt{u_1}}, & \frac{1}{24} \frac{u_2^2 u_5}{u_1^{3/2} u_3} - \frac{1}{2} \sqrt{u_1} u_6 \\ 0 & \frac{u_3}{\sqrt{u_1}}, & -\frac{u_5}{\sqrt{u_1}}, & \frac{1}{2} \frac{u_2 u_5 - u_3 u_4}{u_1^{3/2}} \\ 0 & 0 & \frac{\sqrt{u_1}}{u_3}, & -\frac{1}{2} \frac{u_2}{\sqrt{u_1} u_3} \\ 0 & 0 & 0 & \frac{1}{\sqrt{u_1}} \end{pmatrix}.$$

- ii) *The coframe  $(\theta^1, \theta^2, \theta^3, \theta^4, \Omega_1, \Omega_2, \Omega_3, \Omega_4, \Omega_5, \Omega_6)$  on  $P^c$ , which defines the  $\mathfrak{sp}(4, \mathbb{R})$ -valued Cartan normal connection  $\widehat{\omega}^c$  on  $P^c$  by*

$$(17) \quad \widehat{\omega}^c = \begin{pmatrix} \frac{1}{2} \Omega_1 & \frac{1}{2} \Omega_2 & -\frac{1}{2} \Omega_4 & -\frac{1}{4} \Omega_6 \\ \theta^4 & \Omega_3 - \frac{1}{2} \Omega_1 & -\Omega_5 & -\frac{1}{2} \Omega_4 \\ \theta^2 & \theta^3 & \frac{1}{2} \Omega_1 - \Omega_3 & -\frac{1}{2} \Omega_2 \\ 2\theta^1 & \theta^2 & -\theta^4 & -\frac{1}{2} \Omega_1 \end{pmatrix}.$$

Two 3rd order ODEs  $y''' = F(x, y, y', y'')$  and  $y''' = \bar{F}(x, y, y', y'')$  are locally contact equivalent if and only if their associated Cartan connections are locally diffeomorphic, that is there exists a local bundle diffeomorphism  $\Phi: \bar{P}^c \supset \bar{\mathcal{O}} \rightarrow \mathcal{O} \subset P^c$  such that

$$\Phi^* \widehat{\omega}^c = \overline{\widehat{\omega}^c}.$$

The connection  $\widehat{\omega}^c$  has the following explicit form. Let  $(x, y, p, q, u_1, u_2, u_3, u_4, u_5, u_6), (x^i, u_\mu)$  for short, be a local coordinate system in  $P^c$ , which is compatible with the local trivialization  $P^c = H_6 \times J^2$ , that is  $(x^i) = (x, y, p, q)$  are coordinates in  $J^2$  and  $(u_\mu)$  are coordinates in  $H_6$  as in (16). Then the value of  $\widehat{\omega}^c$  at the point  $(x^i, u_\mu)$  in  $P^c$  is given by

$$\widehat{\omega}^c(x^i, u_\mu) = u^{-1} \omega^c u + u^{-1} du$$

where  $u$  denotes the matrix (16) and

$$\omega^c = \begin{pmatrix} \frac{1}{2}\Omega_1^0 & \frac{1}{2}\Omega_2^0 & -\frac{1}{2}\Omega_4^0 & -\frac{1}{4}\Omega_6^0 \\ \omega^4 & \Omega_3^0 - \frac{1}{2}\Omega_1^0 & -\Omega_5^0 & -\frac{1}{2}\Omega_4^0 \\ \omega^2 & \tilde{\omega}^3 & \frac{1}{2}\Omega_1^0 - \Omega_3^0 & -\frac{1}{2}\Omega_2^0 \\ 2\omega^1 & \omega^2 & -\omega^4 & -\frac{1}{2}\Omega_1^0 \end{pmatrix}$$

is the connection  $\widehat{\omega}^c$  calculated at the point  $(x^i, u_1 = 1, u_2 = 0, u_3 = 1, u_4 = 0, u_5 = 0, u_6 = 0)$ . The forms  $\omega^1, \omega^2, \tilde{\omega}^3, \omega^4$  read

$$\begin{aligned} \omega^1 &= dy - p dx, \\ \omega^2 &= dp - q dx, \\ \tilde{\omega}^3 &= dq - F dx - \frac{1}{3}F_q(dp - q dx) + K(dy - p dx), \\ \omega^4 &= dx. \end{aligned} \tag{18}$$

The forms  $\Omega_1^0, \dots, \Omega_6^0$  read

$$\begin{aligned} \Omega_1^0 &= -K_q \omega^1, \\ \Omega_2^0 &= \left(\frac{1}{3}W_q + L\right) \omega^1 - K_q \omega^2 - K \omega^4, \\ \Omega_3^0 &= -K_q \omega^1 + \frac{1}{6}F_{qq} \omega^2 + \frac{1}{3}F_q \omega^4, \\ \Omega_4^0 &= -\left(\frac{1}{3}W_{qq} + L_q\right) \omega^1 + \frac{1}{2}K_{qq} \omega^2, \\ \Omega_5^0 &= \frac{1}{2}K_{qq} \omega^1 - \frac{1}{6}F_{qqq} \omega^2 - \frac{1}{6}F_{qq} \omega^4, \\ \Omega_6^0 &= \left(\frac{1}{3}\mathcal{D}(W_{qq}) - \frac{4}{3}W_{qp} - \frac{1}{3}F_q W_{qq} + \frac{1}{3}F_{qqq}W + M\right) \omega^1 + \\ &\quad + \frac{1}{3}(F_{qqy} - F_{qqq}K - W_{qq}) \omega^2 - K_{qq} \tilde{\omega}^3 + \\ &\quad + \left(\frac{2}{3}F_{qy} - \frac{1}{3}F_{qq}K - 2L - \frac{4}{3}W_q\right) \omega^4. \end{aligned} \tag{19}$$

**3.2. Proof of theorem 3.1.** We prove theorem 3.1 by repeating Chern's construction supplemented later in [43].

On the bundle  $G_c \times J^2$  there are four fixed, well defined one-forms  $(\theta^1, \theta^2, \theta^3, \theta^4)$ , the components of the canonical  $\mathbb{R}^4$ -valued form  $\theta$  existing on the frame bundle of  $J^2$ . Let  $(x)$  denote  $x, y, p, q$  and  $(g)$  be coordinates in  $G_c$  given by (12). Let us choose a coordinate system  $(x, g)$  on  $G_c \times J^2$  compatible with the local trivialization. Then  $\theta^i$  at the point  $(x, g^{-1})$  read

$$\begin{aligned} \theta^1 &= u_1 \omega^1, \\ \theta^2 &= u_2 \omega^1 + u_3 \omega^2, \\ \theta^3 &= u_4 \omega^1 + u_5 \omega^2 + u_6 \omega^3, \\ \theta^4 &= u_8 \omega^1 + u_9 \omega^2 + u_7 \omega^4. \end{aligned} \tag{20}$$

We seek a bundle on which  $\theta^i$  are supplemented to a coframe by certain new one-forms  $\Omega_\mu$  chosen in a well-defined geometric manner.

Step 1. We calculate the exterior derivatives of  $\theta^i$  on  $G_c \times J^2$

$$\begin{aligned} d\theta^1 &= \alpha_1 \wedge \theta^1 + T_{jk}^1 \theta^j \wedge \theta^k, \\ d\theta^2 &= \alpha_2 \wedge \theta^1 + \alpha_3 \wedge \theta^2 + T_{jk}^2 \theta^j \wedge \theta^k, \\ d\theta^3 &= \alpha_4 \wedge \theta^1 + \alpha_5 \wedge \theta^2 + \alpha_6 \wedge \theta^3 + T_{jk}^3 \theta^j \wedge \theta^k, \\ d\theta^4 &= \alpha_8 \wedge \theta^1 + \alpha_9 \wedge \theta^2 + \alpha_7 \wedge \theta^4 + T_{jk}^4 \theta^j \wedge \theta^k, \end{aligned} \tag{21}$$

where  $\alpha_\mu$  are the entries of the matrix  $dg_k^i \cdot g_j^{-1k}$  and  $T_{jk}^i$  are some functions on  $G_c \times J^2$ . Next we collect  $T_{jk}^i \theta^j \wedge \theta^k$  terms

$$\begin{aligned}
 d\theta^1 &= (\alpha_1 - T_{12}^1 \theta^2 - T_{13}^1 \theta^3 - T_{14}^1 \theta^4) \wedge \theta^1 \\
 &\quad + T_{23}^1 \theta^2 \wedge \theta^3 + T_{24}^1 \theta^2 \wedge \theta^4 + T_{34}^1 \theta^3 \wedge \theta^4, \\
 d\theta^2 &= (\alpha_2 - T_{12}^2 \theta^2 - T_{13}^2 \theta^3 - T_{14}^2 \theta^4) \wedge \theta^1 \\
 &\quad + (\alpha_3 - T_{23}^2 \theta^3 - T_{24}^2 \theta^4) \wedge \theta^2 + T_{34}^2 \theta^3 \wedge \theta^4, \\
 d\theta^3 &= (\alpha_4 - T_{12}^3 \theta^2 - T_{13}^3 \theta^3 - T_{14}^3 \theta^4) \wedge \theta^1, \\
 &\quad + (\alpha_5 - T_{23}^3 \theta^3 - T_{24}^3 \theta^4) \wedge \theta^2 + (\alpha_6 - T_{34}^3 \theta^4) \wedge \theta^3 \\
 d\theta^4 &= (\alpha_8 - T_{12}^4 \theta^2 - T_{13}^4 \theta^3 - T_{14}^4 \theta^4) \wedge \theta^1, \\
 &\quad + (\alpha_9 - T_{23}^4 \theta^3 - T_{24}^4 \theta^4) \wedge \theta^2 + (\alpha_7 + T_{34}^4 \theta^3) \wedge \theta^4
 \end{aligned} \tag{22}$$

and introduce new 1-forms  $\pi_\mu$  substituting the collected terms. Eq. (21) now read

$$\begin{aligned}
 d\theta^1 &= \pi_1 \wedge \theta^1 + \frac{u_1}{u_3 u_7} \theta^4 \wedge \theta^2, \\
 d\theta^2 &= \pi_2 \wedge \theta^1 + \pi_3 \wedge \theta^2 + \frac{u_3}{u_6 u_7} \theta^4 \wedge \theta^3, \\
 d\theta^3 &= \pi_4 \wedge \theta^1 + \pi_5 \wedge \theta^2 + \pi_6 \wedge \theta^3, \\
 d\theta^4 &= \pi_8 \wedge \theta^1 + \pi_9 \wedge \theta^2 + \pi_7 \wedge \theta^4,
 \end{aligned} \tag{23}$$

since  $T_{23}^1 = T_{34}^1 = 0$ ,  $T_{24}^1 = -u_3 u_7 / u_1$  and  $T_{34}^2 = -u_3 / (u_6 u_7)$ .

The equations (23) resemble structural equations for a linear connection very much, however, here  $\pi$  is not a linear connection since it does not transform according to  $R_u^* \pi = \text{Ad}(u^{-1}) \pi$  along fibres of  $G_c \times J^2$ . We may think of  $\pi$  as a connection in a broader meaning, that is a horizontal distribution on  $G_c \times J^2$ , which is not necessarily right-invariant. Keeping this in mind we will refer to  $T_{jk}^i$  as torsion. Thus  $\pi$  is a connection chosen by the demand that its torsion is ‘minimal’, i.e. possesses as few terms as possible. The forms  $\pi_\mu$ , which are candidates for the sought forms  $\Omega_\mu$ , are not uniquely defined by equations (23), for example the gauge  $\pi_1 \rightarrow \pi_1 + f \theta^1$  leaves (23) unchanged. Therefore our connection is not uniquely defined by its torsion.

Step 2. We reduce the bundle  $G_c \times J^2$ . We choose its subbundle, say  $P^{(1)}$ , characterized by the property that the torsion coefficients are constant on it. We choose  $P^{(1)}$  such that  $T_{24}^1 = -1$ ,  $T_{34}^2 = -1$  on it. Thus  $P^{(1)}$  is defined by

$$u_6 = \frac{u_3^2}{u_1}, \quad u_7 = \frac{u_1}{u_3}. \tag{24}$$

It is known [25, 44] that such a reduction preserves the equivalence, in other words, two bundles are equivalent if and only if their respective reductions are. Here  $P^{(1)}$  has the seven-dimensional structural group

$$G_c^{(1)} = \begin{pmatrix} u_1 & 0 & 0 & 0 \\ u_2 & u_3 & 0 & 0 \\ u_4 & u_5 & \frac{u_3^2}{u_1} & 0 \\ u_8 & u_9 & 0 & \frac{u_1}{u_3} \end{pmatrix}.$$

Step 3. Next we pull-back  $\theta^i$  and  $\pi_\mu$  to  $P^{(1)}$ . But the new structural group  $G_c^{(1)}$  is a seven-dimensional subgroup of  $G_c$ , so  $(\theta^1, \dots, \theta^4, \pi_1, \dots, \pi_9)$  of (23) is not a coframe on  $P^{(1)}$  any longer, since

$$\pi_6 = 2\pi_3 - \pi_1 \pmod{(\theta^i)}, \quad \pi_7 = \pi_1 - \pi_3 \pmod{(\theta^i)}.$$

Taking this into account we recalculate (23) and gather the torsion terms. We choose the new connection

$$\pi = \begin{pmatrix} \pi_1 & 0 & 0 & 0 \\ \pi_2 & \pi_3 & 0 & 0 \\ \pi_4 & \pi_5 & 2\pi_3 - \pi_1 & 0 \\ \pi_8 & \pi_9 & 0 & \pi_1 - \pi_3 \end{pmatrix}$$

so that its torsion is minimal again.

$$\begin{aligned} d\theta^1 &= \pi_1 \wedge \theta^1 + \theta^4 \wedge \theta^2, \\ d\theta^2 &= \pi_2 \wedge \theta^1 + \pi_3 \wedge \theta^2 + \theta^4 \wedge \theta^3, \\ (25) \quad d\theta^3 &= \pi_4 \wedge \theta^1 + \pi_5 \wedge \theta^2 + (2\pi_3 - \pi_1) \wedge \theta^3 + \left( \frac{3u_5}{u_3} - \frac{3u_2 - u_3 F_q}{u_1} \right) \theta^4 \wedge \theta^3, \\ d\theta^4 &= \pi_8 \wedge \theta^1 + \pi_9 \wedge \theta^2 + (\pi_1 - \pi_3) \wedge \theta^4. \end{aligned}$$

Step 4. We repeat the steps 2. and 3. Firstly we reduce  $P^{(1)}$  to the subbundle  $P^{(2)} \subset P^{(1)}$  defined by the property that the only non-constant torsion coefficient  $T_{34}^3$  in (25) vanishes on it,

$$(26) \quad u_5 = \frac{u_3}{u_1} \left( u_2 - \frac{1}{3} u_3 F_q \right).$$

Next we recalculate connection, re-collect the torsion and make another reduction through the constant torsion condition ( $K$  is defined in (5).)

$$(27) \quad u_4 = \frac{u_3^2}{u_1} K + \frac{u_2^2}{2u_1}.$$

At this stage we have reduced the frame bundle  $G_c \times J^2$  to the nine-dimensional subbundle  $P^{(3)} \rightarrow J^2$ , such that its structural group is the following

$$G_c^{(3)} = \begin{pmatrix} u_1 & 0 & 0 & 0 \\ u_2 & u_3 & 0 & 0 \\ \frac{u_2^2}{u_1} & \frac{u_2 u_3}{u_1} & \frac{u_3^3}{u_1} & 0 \\ u_8 & u_9 & 0 & \frac{u_1}{u_3} \end{pmatrix}$$

and the frame dual to  $(\omega^1, \omega^2, \omega^3 - \frac{1}{3} F_q \omega^2 + K \omega^1, \omega^4)$  belongs to  $P^{(3)}$ . The structural equations on  $P^{(3)}$  read after collecting

$$\begin{aligned} d\theta^1 &= \pi_1 \wedge \theta^1 + \theta^4 \wedge \theta^2, \\ d\theta^2 &= \pi_2 \wedge \theta^1 + \pi_3 \wedge \theta^2 + \theta^4 \wedge \theta^3, \\ (28) \quad d\theta^3 &= \pi_2 \wedge \theta^2 + (2\pi_3 - \pi_1) \wedge \theta^3 + \frac{u_3^3}{u_1^3} W \theta^4 \wedge \theta^1, \\ d\theta^4 &= \pi_8 \wedge \theta^1 + \pi_9 \wedge \theta^2 + (\pi_1 - \pi_3) \wedge \theta^4 \end{aligned}$$

with some one-forms  $\pi_1, \pi_2, \pi_3, \pi_8, \pi_9$ . The function  $W$ , defined in (8), is the Wünschmann invariant. Thereby, as Chern observed, third-order ODEs fall into two main contact inequivalent branches: the ODEs satisfying  $W \neq 0$ , and those satisfying  $W = 0$ .

Equations (28) do not still define the forms  $\pi_\mu$  uniquely but only modulo the following transformations

$$\begin{aligned} \pi_1 &\rightarrow \pi_1 + 2t_1 \theta^1, \\ \pi_2 &\rightarrow \pi_2 + t_1 \theta^2, \\ (29) \quad \pi_3 &\rightarrow \pi_3 + t_1 \theta^1, \\ \pi_8 &\rightarrow \pi_8 + t_2 \theta^1 + t_3 \theta^2 + t_1 \theta^4, \end{aligned}$$

$$\pi_9 \rightarrow \pi_9 + t_3\theta^1 + t_4\theta^2.$$

At this point, there is no pattern of further reduction. If  $W = 0$  there are only constant torsion coefficients in (28) and we do not have any conditions to define a subbundle of  $P^{(3)}$ . In these circumstances we prolong  $P^{(3)}$ .

Step 5. Prolongation. On  $P^{(3)}$  there is no fixed coframe but only the coframe  $(\theta^1, \theta^2, \theta^3, \theta^4, \pi_1, \pi_2, \pi_3, \pi_8, \pi_9)$  given modulo (29). But ‘a coframe given modulo  $G$ ’ is a  $G$ -structure on  $P^{(3)}$  and we can deal with this new structure on  $P^{(3)}$  by means of the Cartan method. Let us consider the bundle  $G^{prol} \times P^{(3)}$  then, where

$$G^{prol} = \begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix}$$

reflects the freedom (29) so that the block  $t$  reads

$$\begin{pmatrix} 2t_1 & 0 & 0 & 0 \\ 0 & t_1 & 0 & 0 \\ t_1 & 0 & 0 & 0 \\ t_2 & t_3 & 0 & t_1 \\ t_3 & t_4 & 0 & 0 \end{pmatrix}.$$

On  $P^{(3)} \times G^{prol}$  there exist nine fixed one-forms  $\theta^1, \theta^2, \theta^3, \theta^4, \Pi_1, \Pi_2, \Pi_3, \Pi_8, \Pi_9$ , given by

$$\begin{pmatrix} \theta^i \\ \Pi_\mu \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix} \begin{pmatrix} \theta^i \\ \pi_\mu \end{pmatrix},$$

which is the canonical one-form on  $G^{prol} \times P^{(3)} \rightarrow P^{(3)}$ .

Step 6. Now we apply the method of reductions to the above structure on  $G^{prol} \times P^{(3)}$ . We calculate the exterior derivatives of  $(\theta^i, \Pi_\mu)$ . The derivatives of  $\theta^i$  take the form of (28) with  $\pi_\mu$  replaced by  $\Pi_\mu$ . The derivatives of  $\Pi_\mu$ , after collecting and introducing 1-forms  $\Lambda_I$  containing  $dt_I$ ,  $I = 1, 2, 3, 4$ , read

$$\begin{aligned} d\Pi_1 &= \Lambda_1 \wedge \theta^1 + \Pi_8 \wedge \theta^2 - \Pi_2 \wedge \theta^4, \\ d\Pi_2 &= \frac{1}{2}\Lambda_1 \wedge \theta^2 - \Pi_1 \wedge \Pi_2 - \Pi_2 \wedge \Pi_3 + \Pi_8 \wedge \theta^3 + \frac{u_3^3}{u_1^3} W \Pi_9 \wedge \theta^1 \\ (30) \quad &+ 2f_1\theta^1 \wedge \theta^3 + f_4\theta^1 \wedge \theta^4 + f_2\theta^2 \wedge \theta^3 + f_5\theta^2 \wedge \theta^4, \\ d\Pi_3 &= \frac{1}{2}\Lambda_1 \wedge \theta^1 + \Pi_8 \wedge \theta^2 + \Pi_9 \wedge \theta^3 + f_1\theta^1 \wedge \theta^2 + f_2\theta^1 \wedge \theta^3 + f_5\theta^1 \wedge \theta^4 + f_3\theta^2 \wedge \theta^3, \\ d\Pi_8 &= \Lambda_2 \wedge \theta^1 + \Lambda_3 \wedge \theta^1 + \frac{1}{2}\Lambda_1 \wedge \theta^4 + \Pi_9 \wedge \Pi_2 + \Pi_8 \wedge \Pi_3 + f_2\theta^3 \wedge \theta^4, \\ d\Pi_9 &= \Lambda_3 \wedge \theta^1 + \Lambda_4 \wedge \theta^2 + \Pi_1 \wedge \Pi_9 - 2\Pi_3 \wedge \Pi_9 + \Pi_8 \wedge \theta^4 - f_1\theta^1 \wedge \theta^4 + f_3\theta^3 \wedge \theta^4. \end{aligned}$$

where  $f_1, f_2, f_3, f_4, f_5$  are functions on  $G^{prol} \times P^{(3)}$ . We choose the subbundle  $P^c$  of  $G^{prol} \times P^{(3)}$  by the condition that  $f_1, f_2, f_3$  are equal to zero on  $P^c$ . This is done by appropriate specifying of parameters  $t_2, t_3, t_4$  as functions of  $(x, y, p, q, u_1, u_2, u_3, u_8, u_9, t_1)$ . We skip writing these complicated formulae. The structural equations on  $P^c$  read

$$\begin{aligned} d\theta^1 &= \Pi_1 \wedge \theta^1 + \theta^4 \wedge \theta^2, \\ d\theta^2 &= \Pi_2 \wedge \theta^1 + \Pi_3 \wedge \theta^2 + \theta^4 \wedge \theta^3, \\ d\theta^3 &= \Pi_2 \wedge \theta^2 + (2\Pi_3 - \Pi_1) \wedge \theta^3 + A\theta^4 \wedge \theta^1, \\ d\theta^4 &= \Pi_8 \wedge \theta^1 + \Pi_9 \wedge \theta^2 + (\Pi_1 - \Pi_2) \wedge \theta^4, \\ d\Pi_1 &= \Lambda_1 \wedge \theta^1 + \Pi_8 \wedge \theta^2 - \Pi_2 \wedge \theta^4, \\ d\Pi_2 &= (\Pi_3 - \Pi_1) \wedge \Pi_2 + A\Pi_9 \wedge \theta^1 + \frac{1}{2}\Lambda_1 \wedge \theta^2 + \Pi_8 \wedge \theta^3 + B\theta^1 \wedge \theta^4 + C\theta^2 \wedge \theta^4, \\ (31) \quad d\Pi_3 &= \frac{1}{2}\Lambda_1 \wedge \theta^1 + \Pi_8 \wedge \theta^2 + \Pi_9 \wedge \theta^3 + C\theta^1 \wedge \theta^4, \end{aligned}$$

$$\begin{aligned}
d\Pi_8 &= \Pi_9 \wedge \Pi_2 + \Pi_8 \wedge \Pi_3 - 2C \Pi_9 \wedge \theta^1 + \frac{1}{2} \Lambda_1 \wedge \theta^4 + D \theta^1 \wedge \theta^2 + 2E \theta^1 \wedge \theta^3 \\
&\quad + G \theta^1 \wedge \theta^4 + H \theta^2 \wedge \theta^3 + J \theta^2 \wedge \theta^4, \\
d\Pi_9 &= (\Pi_1 - 2\Pi_3) \wedge \Pi_9 + \Pi_8 \wedge \theta^4 + E \theta^1 \wedge \theta^2 + H \theta^1 \wedge \theta^3 + J \theta^1 \wedge \theta^4 + L \theta^2 \wedge \theta^3, \\
d\Lambda_1 &= \Lambda_1 \wedge \Pi_1 + 2\Pi_8 \wedge \Pi_2 + 2C \Pi_8 \wedge \theta^1 - 2C \Pi_9 \wedge \theta^2 - A \Pi_9 \wedge \theta^4 + \widetilde{M} \theta^1 \wedge \theta^2 \\
&\quad + 2(D + AL) \theta^1 \wedge \theta^3 + \widetilde{N} \theta^1 \wedge \theta^4 + 2E \theta^2 \wedge \theta^3 + G \theta^2 \wedge \theta^4
\end{aligned}$$

with certain functions  $A, B, C, D, E, F, G, H, J, L, \widetilde{M}, \widetilde{N}$  on  $P^c$ .

Above structural equations *uniquely define the only remaining auxiliary form*  $\Lambda_1$ . In this manner we constructed the bundle  $P^c \rightarrow J^2$  and the fixed coframe associated to the ODEs modulo contact transformations.

**3.2.1. Cartan normal connection from Tanaka's theory.** The above coframe is not fully satisfactory from the geometric point of view since it does not transform equivariantly along the fibres of  $P^c \rightarrow J^2$ .

In order to see this we consider the simplest case, related to the equation  $y''' = 0$ , when all the functions  $A, \dots, \widetilde{N}$  vanish. Then (31) become the Maurer-Cartan equations for the Lie algebra  $\mathfrak{o}(3, 2) \cong \mathfrak{sp}(4, \mathbb{R})$  and  $P^c$  is locally the Lie group  $SP(4, \mathbb{R})$ . The Maurer-Cartan form on  $P^c$  in the four-dimensional defining representation of  $\mathfrak{sp}(4, \mathbb{R})$  is given by

$$\widetilde{\omega} = \begin{pmatrix} \frac{1}{2}\Pi_1 & \frac{1}{2}\Pi_2 & -\frac{1}{2}\Pi_8 & -\frac{1}{4}\Lambda_1 \\ \theta^4 & \Pi_3 - \frac{1}{2}\Pi_1 & -\Pi_9 & -\frac{1}{2}\Pi_8 \\ \theta^2 & \theta^3 & \frac{1}{2}\Pi_1 - \Pi_3 & -\frac{1}{2}\Pi_2 \\ 2\theta^1 & \theta^2 & -\theta^4 & -\frac{1}{2}\Pi_1 \end{pmatrix}.$$

However, this object is not a Cartan connection in a general case, when  $A, \dots, \widetilde{N}$  do not vanish, since its curvature  $\widetilde{K} = d\widetilde{\omega} + \widetilde{\omega} \wedge \widetilde{\omega}$  is not horizontal with respect to the fibration  $P \rightarrow J^2$ , that is the value of  $\widetilde{K}$  on a vector tangent to a fibre of  $P \rightarrow J^2$  is not necessarily zero; for instance  $\widetilde{K}^1_2$  contains the term  $A\Pi_9 \wedge \theta^1$ .

In order to resolve this problem H. Sato and Y. Yoshikawa [43] found the structural equations for the normal connection in this problem by means of the Tanaka theory. We recalculate their result in our notation and give explicit form of the normal connection, which their paper does not contain.

Let  $E^i_j \in \mathfrak{gl}(4, \mathbb{R})$  denotes the matrix whose  $(i, j)$ -component is equal to one and other components equal zero. We introduce the following base in  $\mathfrak{sp}(4, \mathbb{R})$

$$\begin{aligned}
(32) \quad e_1 &= 2E^4_1, & e_2 &= E^3_1 + E^4_2, & e_3 &= E^3_2 \\
e_4 &= E^2_1 - E^4_3, & e_5 &= \frac{1}{2}(E^1_1 - E^2_2 + E^3_3 - E^4_4), & e_6 &= \frac{1}{2}(E^1_2 - E^3_4), \\
e_7 &= E^2_2 - E^3_3, & e_8 &= -\frac{1}{2}(E^1_3 + E^4_2), & e_9 &= -E^2_3, \\
e_{10} &= -\frac{1}{4}E^1_4.
\end{aligned}$$

In this base the form  $\widetilde{\omega}$  is given by

$$\widetilde{\omega} = \theta^1 e_1 + \theta^2 e_2 + \theta^3 e_3 + \theta^4 e_4 + \Pi_1 e_5 + \Pi_2 e_6 + \Pi_3 e_7 + \Pi_8 e_8 + \Pi_9 e_9 + \Lambda_1 e_{10}.$$

The normal Cartan connection  $\widehat{\omega}^c$  which we seek reads

$$\widehat{\omega}^c = \theta^1 e_1 + \theta^2 e_2 + \theta^3 e_3 + \theta^4 e_4 + \Omega_1 e_5 + \Omega_2 e_6 + \Omega_3 e_7 + \Omega_4 e_8 + \Omega_5 e_9 + \Omega_6 e_{10},$$

where the forms  $\Omega_\mu$ , unknown yet, are given by

$$\Omega_1 = \Pi_1 + a_i \theta^i, \quad \Omega_2 = \Pi_2 + b_i \theta^i, \quad \Omega_3 = \Pi_3 + c_i \theta^i,$$



$$\Omega_4 = \Pi_8 + f_i \theta^i, \quad \Omega_5 = \Pi_9 + g_i \theta^i, \quad \Omega_6 = \Lambda_1 + h_i \theta^i$$

and the functions  $a_i, b_i, c_i, f_i, g_i, h_i$  are to be found from the normality conditions of definition 2.6. The algebra  $\mathfrak{sp}(4, \mathbb{R})$  has the following grading

$$(33) \quad \mathfrak{sp}(4, \mathbb{R}) = \mathfrak{g}_{-3} \oplus \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \mathfrak{g}_2 \oplus \mathfrak{g}_3,$$

where

$$\begin{aligned} \mathfrak{g}_{-3} &= \langle e_1 \rangle, & \mathfrak{g}_{-2} &= \langle e_2 \rangle, & \mathfrak{g}_{-1} &= \langle e_3, e_4 \rangle, \\ \mathfrak{g}_0 &= \langle e_5, e_7 \rangle, & \mathfrak{g}_1 &= \langle e_6, e_9 \rangle, & \mathfrak{g}_2 &= \langle e_8 \rangle, & \mathfrak{g}_3 &= \langle e_{10} \rangle \end{aligned}$$

and  $\mathfrak{h} = \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \mathfrak{g}_2 \oplus \mathfrak{g}_3 = \langle e_5, \dots, e_{10} \rangle$ . Let lower case Latin indices range from 1 to 4 and upper case Latin indices range from 1 to 10 throughout this section.

The structure function  $\kappa$  of  $\widehat{\omega}^c$ , defined in (15), decomposes into

$$\kappa = \frac{1}{2} \kappa_{ij}^I e_I \otimes e^i \wedge e^j,$$

where  $(e^I)$  denotes the basis dual to  $(e_I)$  and  $\kappa_{ij}^I = \kappa_{[ij]}^I$  are functions. Condition i) of definition 2.6 reads

$$\kappa_{23}^1 = 0, \quad \kappa_{24}^1 = 0, \quad \kappa_{34}^1 = 0, \quad \kappa_{34}^2 = 0.$$

We read structural constants  $[e_I, e_J] = c_{IJ}^K e_K$  for  $\mathfrak{sp}(4, \mathbb{R})$ , compute the Killing form  $B_{IJ}$  and its inverse  $B^{IJ}$ . The operator  $\partial^*: \text{Hom}(\wedge^2 \mathfrak{m}, \mathfrak{g}) \rightarrow \text{Hom}(\mathfrak{m}, \mathfrak{g})$  acts as follows

$$\partial^*(e_I \otimes e^i \wedge e^j) = \left( 2\delta_m^{[i} B^{j]K} c_{KI}^L - \delta_I^L c_{Km}^{[i} B^{j]K} \right) e_L \otimes e^m.$$

We apply  $\partial^*$  to the basis  $(e_I \otimes e^i \wedge e^j)$  of  $\text{Hom}^1(\wedge^2 \mathfrak{m}, \mathfrak{g})$  and find that the condition ii) of definition 2.6,  $\partial^* \kappa = 0$ , is equivalent to vanishing of the following combinations of  $\kappa_{ij}^I$ :

$$\begin{aligned} &\kappa_{14}^1, \kappa_{14}^2, \kappa_{24}^2, \kappa_{24}^3, \kappa_{34}^3, \kappa_{23}^4, \kappa_{12}^1 + \kappa_{13}^2 + \kappa_{24}^2, 2\kappa_{12}^1 - \kappa_{24}^4 - \kappa_{34}^5, \\ &\kappa_{12}^1 - \kappa_{23}^3 - \kappa_{34}^7, \kappa_{13}^1 + \kappa_{23}^2, \kappa_{13}^1 - \kappa_{34}^4, \kappa_{12}^2 + \kappa_{14}^4 + \kappa_{24}^5, \\ &\kappa_{12}^2 - \kappa_{13}^3 - \kappa_{24}^7, \kappa_{12}^2 + \kappa_{24}^5 - \kappa_{34}^6 - \kappa_{24}^7, \kappa_{13}^2 + \kappa_{23}^3 + \kappa_{34}^5 - \kappa_{34}^7, \\ &\kappa_{23}^2 + \kappa_{34}^4, \kappa_{12}^3 - \kappa_{14}^5 + \kappa_{24}^6 + \kappa_{14}^7, \kappa_{12}^4 - \kappa_{13}^5 + 2\kappa_{13}^7 + \kappa_{24}^9, \\ &\kappa_{12}^4 - \kappa_{23}^6 - \kappa_{34}^8 - \kappa_{24}^9, \kappa_{13}^4 + \kappa_{23}^7 - \kappa_{34}^9, \kappa_{14}^4 + \kappa_{34}^6 + \kappa_{24}^7, \\ &\kappa_{24}^4 - \kappa_{34}^5 + 2\kappa_{34}^7, 2\kappa_{12}^5 - 2\kappa_{24}^8 - \kappa_{34}^{10}, \kappa_{13}^5 + \kappa_{23}^6 - \kappa_{34}^8, \kappa_{14}^5 + \kappa_{24}^6, \\ &\kappa_{23}^5 - 2\kappa_{23}^7 - \kappa_{34}^9, 2\kappa_{12}^6 + 2\kappa_{14}^8 + \kappa_{24}^{10}, \kappa_{13}^6 + \kappa_{12}^7 + \kappa_{24}^8 + \kappa_{14}^9. \end{aligned}$$

Next, we calculate the curvature  $\widehat{K}^c = d\widehat{\omega}^c + \widehat{\omega}^c \wedge \widehat{\omega}^c$ , find the components of the structure function and put them into the normality conditions. These are only satisfied if all the functions  $a_i, b_i, c_i, e_i, f_i, g_i$  vanish except for  $c_1, h_1, h_2, h_3, h_4$  which are arbitrary and

$$a_1 = 2c_1, \quad b_1 = \frac{2}{3}C, \quad b_2 = c_1, \quad f_1 = \frac{2}{3}J, \quad f_4 = c_1,$$

where  $C$  and  $J$  are the functions in (31). Finally, we obtain from the  $e_1$ -component of the Bianchi identity  $d\widehat{K}^c = \widehat{K}^c \wedge \widehat{\omega}^c - \widehat{\omega}^c \wedge \widehat{K}^c$  that

$$c_1 = 0, \quad h_1 = \frac{4}{3}G - \frac{2}{3}\widetilde{X}_4(J), \quad h_2 = \frac{2}{3}J, \quad h_3 = 0, \quad h_4 = -\frac{2}{3}C,$$

where  $\widehat{X}_4$  is the vector field in the frame  $(\widetilde{X}_1, \widetilde{X}_2, \widetilde{X}_3, \widetilde{X}_4, \widetilde{X}_5, \widetilde{X}_6, \widetilde{X}_7, \widetilde{X}_8, \widetilde{X}_9, \widetilde{X}_{10})$  dual to  $(\theta^1, \theta^1, \theta^1, \theta^1, \Pi_1, \Pi_2, \Pi_3, \Pi_8, \Pi_9, \Lambda_1)$ . The normal connection  $\widehat{\omega}^c$  has been constructed. The last thing we must do is renaming the coordinates  $u_8 \rightarrow u_4$ ,  $u_9 \rightarrow u_5$  and choosing  $u_6$  appropriately, so that formulae (16) – (19) hold. This finishes the proof of theorem 3.1.

### 3.3. Ten-dimensional bundle $P^c$ .

3.3.1. *Curvature.* We turn to discussion of consequences of theorem 3.1. The curvature

$$\widehat{K}^c = d\widehat{\omega}^c + \widehat{\omega}^c \wedge \widehat{\omega}^c$$

is given by the non-constant terms in the structural equations for the coframe  $\theta^1, \dots, \Omega_6$ .

$$\begin{aligned}
 d\theta^1 &= \Omega_1 \wedge \theta^1 + \theta^4 \wedge \theta^2, \\
 d\theta^2 &= \Omega_2 \wedge \theta^1 + \Omega_3 \wedge \theta^2 + \theta^4 \wedge \theta^3, \\
 d\theta^3 &= \Omega_2 \wedge \theta^2 + (2\Omega_3 - \Omega_1) \wedge \theta^3 + \mathbf{A}_2^c \theta^2 \wedge \theta^1 + \mathbf{A}_1^c \theta^4 \wedge \theta^1, \\
 d\theta^4 &= \Omega_4 \wedge \theta^1 + \Omega_5 \wedge \theta^2 + (\Omega_1 - \Omega_3) \wedge \theta^4, \\
 d\Omega_1 &= \Omega_6 \wedge \theta^1 + \Omega_4 \wedge \theta^2 - \Omega_2 \wedge \theta^4, \\
 d\Omega_2 &= (\Omega_3 - \Omega_1) \wedge \Omega_2 + \frac{1}{2} \Omega_6 \wedge \theta^2 + \Omega_4 \wedge \theta^3 + \mathbf{A}_3^c \theta^1 \wedge \theta^2 + \mathbf{A}_4^c \theta^1 \wedge \theta^4, \\
 (34) \quad d\Omega_3 &= \frac{1}{2} \Omega_6 \wedge \theta^1 + \Omega_4 \wedge \theta^2 + \Omega_5 \wedge \theta^3 + \mathbf{A}_5^c \theta^1 \wedge \theta^2 + \mathbf{A}_2^c \theta^1 \wedge \theta^4, \\
 d\Omega_4 &= \Omega_5 \wedge \Omega_2 + \Omega_4 \wedge \Omega_3 + \frac{1}{2} \Omega_6 \wedge \theta^4 + (\mathbf{A}_6^c + \mathbf{B}_2^c) \theta^1 \wedge \theta^2 + 2\mathbf{B}_3^c \theta^1 \wedge \theta^3 \\
 &\quad - \mathbf{A}_3^c \theta^1 \wedge \theta^4 + \mathbf{B}_4^c \theta^2 \wedge \theta^3 \\
 d\Omega_5 &= (\Omega_1 - 2\Omega_3) \wedge \Omega_5 + \Omega_4 \wedge \theta^4 + (\mathbf{A}_7^c + \mathbf{B}_3^c) \theta^1 \wedge \theta^2 + \mathbf{B}_4^c \theta^1 \wedge \theta^3 \\
 &\quad - \mathbf{A}_5^c \theta^1 \wedge \theta^4 + \mathbf{B}_1^c \theta^2 \wedge \theta^3, \\
 d\Omega_6 &= \Omega_6 \wedge \Omega_1 + 2\Omega_4 \wedge \Omega_2 + \mathbf{C}_1^c \theta^1 \wedge \theta^2 + 2\mathbf{B}_2^c \theta^1 \wedge \theta^3 + \mathbf{A}_8^c \theta^1 \wedge \theta^4 + 2\mathbf{B}_3^c \theta^2 \wedge \theta^3,
 \end{aligned}$$

where  $\mathbf{A}_1^c, \dots, \mathbf{A}_8^c, \mathbf{B}_1^c, \dots, \mathbf{B}_4^c, \mathbf{C}_1^c$  are functions on  $P^c$ .

The functions  $\mathbf{A}_1^c, \dots, \mathbf{C}_1^c$  are contact relative invariants of the underlying ODE and the full set of contact invariants can be constructed by consecutive differentiation of  $\mathbf{A}_1^c, \dots, \mathbf{C}_1^c$  with respect to the frame  $(X_1, X_2, X_3, X_4, X_5, X_6, X_7, X_8, X_9, X_{10})$  dual to  $(\theta^1, \theta^2, \theta^3, \theta^4, \Omega_1, \Omega_2, \Omega_3, \Omega_4, \Omega_5, \Omega_6)$ , see [42]. Utilizing the identities  $d^2\Omega_\mu = 0$  we compute the exterior derivatives of  $\mathbf{A}_i^c, \mathbf{B}_j^c, \mathbf{C}_1^c$ , for instance

$$d\mathbf{B}_1^c = X_1(\mathbf{B}_1^c)\theta^1 + X_2(\mathbf{B}_2^c)\theta^2 + X_3(\mathbf{B}_3^c)\theta^3 - 2\mathbf{B}_4^c\theta^4 + 2\mathbf{B}_1^c\Omega_1 - 5\mathbf{B}_1^c\theta^3.$$

From these formulae it follows that i)  $\mathbf{A}_2^c, \dots, \mathbf{A}_8^c$  express by the coframe derivatives of  $\mathbf{A}_1^c$ , ii)  $\mathbf{B}_2^c, \dots, \mathbf{B}_4^c$  express by coframe derivatives of  $\mathbf{B}_1^c$  and iii)  $\mathbf{C}_1^c$  is a function of coframe derivatives of both  $\mathbf{A}_1^c$  and  $\mathbf{B}_1^c$ . Hence we have

**Corollary 3.2.** *There are two basic contact relative invariants<sup>4</sup> for third order ODEs:*

$$\mathbf{A}_1^c = \frac{u_3^3}{u_1^3} W \quad \mathbf{B}_1^c = \frac{u_1^2}{6u_3^5} F_{qqqq}.$$

*All other invariants can be derived from them by consecutive differentiation with respect to the dual frame  $(X_i)$ .*

The simplest case, in which all the contact invariants  $\mathbf{A}_1^c, \dots, \mathbf{C}_1^c$  vanish corresponds to  $W = F_{qqqq} = 0$  and is characterized by

**Corollary 3.3.** *For a third-order ODE  $y''' = F(x, y, y', y'')$  the following conditions are equivalent.*

- i) *The ODE is contact equivalent to  $y''' = 0$ .*
- ii) *It satisfies the conditions  $W = 0$ , and  $F_{qqqq} = 0$ .*
- iii) *It has the  $\mathfrak{o}(3, 2)$  algebra of contact symmetry generators.*

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<sup>4</sup>This property means in the language of Tanaka's theory that curvature of a normal connection is generated by its harmonic part.

3.3.2. *Structure of  $P^c$ .* The manifold  $P^c$  is endowed with threefold geometry of principal bundle over the second jet space  $J^2$ , the first jet space  $J^1$  and the solution space  $\mathcal{S}$ . We discuss these structures consecutively. Let us remind that  $(X_1, X_2, X_3, X_4, X_5, X_6, X_7, X_8, X_9, X_{10})$  denotes the dual frame to  $(\theta^1, \theta^2, \theta^3, \theta^4, \Omega_1, \Omega_2, \Omega_3, \Omega_4, \Omega_5, \Omega_6)$ .

First bundle structure,  $H_6 \rightarrow P \rightarrow J^2$ , has been already introduced explicitly in theorem 3.1. Here we only show that it is actually defined by the coframe, since it can be recovered from its structural equations. Indeed, we see from (34) that the ideal spanned by  $\theta^1, \theta^2, \theta^3, \theta^4$  is closed

$$d\theta^i \wedge \theta^1 \wedge \theta^2 \wedge \theta^3 \wedge \theta^4 = 0 \quad \text{for} \quad i = 1, 2, 3, 4,$$

and it follows that its annihilated distribution  $\langle X_5, X_6, X_7, X_8, X_9, X_{10} \rangle$  is integrable. Maximal integral leaves of this distribution are locally fibres of the projection  $P^c \rightarrow J^2$ . Furthermore, the commutation relations of these vector fields are isomorphic to the commutation relations of the six-dimensional group  $H_6$ , hence we can *define* the fundamental vector fields associated to the action  $H_6$  on  $P^c$   $X_5, \dots, X_{10}$ .

In order to explain how  $P^c$  is the bundle  $CO(2, 1) \ltimes \mathbb{R}^3 \rightarrow P^c \rightarrow \mathcal{S}$  let us first describe the solution space  $\mathcal{S}$  itself. On  $J^2$  there is the congruence of solutions of the ODE. A family of solutions passing through sufficiently small open set in  $J^2$  is given by the mapping

$$(x; c_1, c_2, c_3) \mapsto (x, f(x; c_1, c_2, c_3), f_x(x; c_1, c_2, c_3), f_{xx}(x; c_1, c_2, c_3)),$$

where  $y = f(x; c_1, c_2, c_3)$  is the general solution to  $y''' = F(x, y, y', y'')$  and  $(c_1, c_2, c_3)$  are constants of integration. Thus a solution can be considered as a point in the three-dimensional real space  $\mathcal{S}$  parameterized by the constants of integration. This space can be endowed with a local structure of differentiable manifold if we *choose* a parametrization  $(c_1, c_2, c_3) \mapsto f(x; c_1, c_2, c_3)$  of the solutions and admit only sufficiently smooth reparameterizations  $(c_1, c_2, c_3) \mapsto (\tilde{c}_1, \tilde{c}_2, \tilde{c}_3)$  of the constants. We always assume that  $\mathcal{S}$  is locally a manifold. Since  $J^2$  is a bundle over  $\mathcal{S}$  so is  $P^c$  and the fibres of the projection  $P^c \rightarrow \mathcal{S}$  are annihilated by the closed ideal  $\langle \theta^1, \theta^2, \theta^3 \rangle$ . On the fibres there act the vector fields  $X_4, X_5, X_6, X_7, X_8, X_9, X_{10}$ , which form the Lie algebra  $\mathfrak{co}(2, 1) \oplus \mathbb{R}^3$  and thereby define the action of  $CO(2, 1) \ltimes \mathbb{R}^3$  on  $P^c$ .

Apart from the projection  $J^2 \rightarrow \mathcal{S}$  there is also the projection  $J^2 \rightarrow J^1$  which takes the second jet  $(x, y, p, q)$  of a curve into its first jet  $(x, y, p)$ . It gives rise to the third bundle structure,  $H_7 \rightarrow P^c \rightarrow J^1$ . Here the tangent distribution is  $\langle X_3, X_5, X_6, X_7, X_8, X_9, X_{10} \rangle$  and it defines the action of a seven-dimensional group  $H_7$  which of course contains  $H_6$ .

It appears that, under some conditions,  $\hat{\omega}_c$  is not only a Cartan connection over  $J^2$  but over  $\mathcal{S}$  or  $J^1$  also.

### 3.4. Conformal geometry on solution space.

3.4.1. *Normal conformal connection.* We remind the notion of normal conformal connection. Consider  $\mathbb{R}^n$  with coordinates  $(x^\mu)$ ,  $\mu = 1, \dots, n$  equipped with the flat metric  $g = g_{\mu\nu} dx^\mu \otimes dx^\nu$  of the signature  $(k, l)$ ,  $n = k + l$ . The group  $Conf(k, l)$  of conformal symmetries of  $g_{\mu\nu}$  consists of

- i) the subgroup  $CO(k, l) = \mathbb{R} \times O(k, l)$  containing the group  $O(k, l)$  of isometries of  $g$  and the dilatations,
- ii) the subgroup  $\mathbb{R}^n$  of translations,
- iii) the subgroup  $\mathbb{R}^n$  of special conformal transformations.

The stabilizer of the origin in  $\mathbb{R}^n$  is the semisimple product of  $CO(k, l) \ltimes \mathbb{R}^n$  of the isometries, the dilatations, and the special conformal transformations. The flat conformal space is the homogeneous space  $Conf(k, l)/CO(k, l) \ltimes \mathbb{R}^n$ . To this space there is associated the flat Cartan connection on the bundle  $CO(k, l) \ltimes \mathbb{R}^n \rightarrow Conf(k, l) \rightarrow \mathbb{R}^n$  with values in the algebra  $\mathfrak{conf}(k, l)$ .

By virtue of the Möbius construction, the group  $Conf(k, l)$  is isomorphic to the orthogonal group  $O(k+1, l+1)$  preserving the metric

$$\begin{pmatrix} 0 & 0 & -1 \\ 0 & g_{\mu\nu} & 0 \\ -1 & 0 & 0 \end{pmatrix}$$

on  $\mathbb{R}^{n+2}$ . This isomorphism gives rise to the following representation of  $\mathfrak{conf}(k, l) \cong \mathfrak{o}(k+1, l+1)$

$$(35) \quad \begin{pmatrix} \phi & g_{\nu\rho}\xi^\rho & 0 \\ v^\mu & \lambda^\mu_\nu & \xi^\mu \\ 0 & g_{\nu\rho}v^\rho & -\phi \end{pmatrix}.$$

Here the vector  $(v^\mu) \in \mathbb{R}^n$  generates the translations, the matrix  $(\lambda^\mu_\nu) \in \mathfrak{o}(k, l)$  generates the isometries,  $\phi$  – the dilatations, and  $(\xi^\mu) \in \mathbb{R}^n$  – the special conformal transformations.

Let us turn to an arbitrary case of a conformal metric  $[g]$  of the signature  $(k, l)$  on a  $n$ -dimensional manifold  $M$ ,  $n = k + l > 2$ . Let us choose a representative  $g$  of  $[g]$  and consider a coframe  $(\omega^\mu)$ , in which  $g = g_{\mu\nu} \omega^\mu \otimes \omega^\nu$  with constant coefficients  $g_{\mu\nu}$ . We calculate the Levi-Civita connection  $\Gamma^\mu_\nu$  for  $g$ , its Ricci tensor  $R_{\mu\nu}$  and the Ricci scalar  $R$ . Next we define the following one-forms

$$P_\nu = \left( \frac{1}{2-n} R_{\nu\rho} + \frac{1}{2(n-1)(n-2)} R g_{\nu\rho} \right) \theta^\rho.$$

Given these objects we build the following  $\mathfrak{conf}(k, l)$ -valued matrix  $\omega^{conf}$  on  $M$

$$(36) \quad \omega^{conf} = \begin{pmatrix} 0 & P_\nu & 0 \\ \theta^\mu & \Gamma^\mu_\nu & g^{\mu\rho} P_\rho \\ 0 & g_{\nu\rho} \theta^\rho & 0 \end{pmatrix}.$$

This is the normal conformal connection on  $M$  in the natural gauge.<sup>5</sup> Consider now the conformal bundle  $CO(k, l) \ltimes \mathbb{R}^n \rightarrow P \rightarrow M$ , and choose a coordinate system  $(h, x)$  on  $P$  compatible with the local triviality  $P \cong CO(k, l) \ltimes \mathbb{R}^n \times M$ , where  $x$  stands for  $(x^\mu)$  in  $M$  and the matrix  $h \in CO(k, l) \ltimes \mathbb{R}^n$  reads

$$h = \begin{pmatrix} e^{-\phi} & e^{-\phi} g_{\nu\rho} \xi^\rho & \frac{1}{2} e^{-\phi} \xi^\rho \xi^\sigma g_{\rho\sigma} \\ 0 & \Lambda^\mu_\nu & \Lambda^\mu_\rho \xi^\rho \\ 0 & 0 & e^\phi \end{pmatrix}, \quad \Lambda^\mu_\rho \Lambda^\nu_\sigma g_{\mu\nu} = g_{\rho\sigma}.$$

The normal conformal connection for  $g$  is the following  $\mathfrak{conf}(k, l)$ -valued one-form on  $P$

$$\hat{\omega}^{conf}(h, x) = h^{-1} \cdot \pi^*(\omega^{conf}(x)) \cdot h + h^{-1} dh.$$

The curvature of the normal conformal connection is as follows

$$\hat{K}^{conf}(h, x) = h^{-1} \cdot \pi^*(K^{conf}(x)) \cdot h,$$

where  $K^{conf}$  is the curvature for  $\omega^{conf}$  on  $M$

$$K^{conf} = \begin{pmatrix} 0 & DP_\nu & 0 \\ 0 & C^\mu_\nu & g^{\mu\rho} DP_\rho \\ 0 & 0 & 0 \end{pmatrix},$$

---

<sup>5</sup>The gauge is natural since we have started from the Levi-Civita connection, not from any Weyl connection for  $g$ , in which case (36) contains a Maxwell potential  $A$ .

and

$$DP_\mu = dP_\mu + P_\nu \wedge \Gamma^\nu_\mu = \frac{1}{2} P_{\mu\nu\rho} \omega^\nu \wedge \omega^\rho.$$

The curvature contains the lowest-order conformal invariants for  $g$ , namely

- For  $n \geq 4$

$$C^\mu_\nu = \frac{1}{2} C^\mu_{\nu\rho\sigma} \omega^\rho \wedge \omega^\sigma$$

is the Weyl conformal tensor, while

$$P_{\mu\nu\rho} = \frac{1}{n-3} \nabla_\sigma C^\sigma_{\mu\nu\rho}$$

is its divergence.

- For  $n = 3$  the Weyl tensor identically vanishes,  $C^\mu_\nu = 0$ , and the lowest-order conformal invariant is the Cotton tensor  $P_{\mu\nu\rho}$ . It has five independent components.

The normality of conformal connections, originally defined by E. Cartan, is the following property. The algebra  $\mathfrak{conf}(k, l) \cong \mathfrak{o}(k+1, l+1)$  is graded:  $\mathfrak{o}(k+1, l+1) = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1$ , where translations are the  $\mathfrak{g}_{-1}$ -part,  $\mathfrak{co}(k, l)$  is the  $\mathfrak{g}_0$ -part and the special conformal transformations are the  $\mathfrak{g}_1$ -part. The normal connection for  $[g]$  is the only conformal connection such that the  $\mathfrak{co}(k, l)$ -part of its curvature,  $C^\mu_\nu = \frac{1}{2} C^\mu_{\nu\rho\sigma} \omega^\rho \wedge \omega^\sigma$ , is traceless:  $C^\rho_{\nu\rho\sigma} = 0$ . Cartan normal conformal connections are normal in the sense of Tanaka.

3.4.2. *Conformal geometries from ODEs.* The following theorem holds.

**Theorem 3.4** (S.-S.Chern). *If a third-order ODE satisfies a contact-invariant condition  $W = 0$ , then it has a Lorentzian conformal geometry  $[g]$  on its solution space  $S$ . Two such geometries constructed from contact equivalent ODEs are diffeomorphic. In the jet coordinates  $[g]$  is represented by*

$$(37) \quad g = (\omega^2)^2 - 2\omega^1\tilde{\omega}^3 = \\ = (dp - qdx)^2 - 2(dy - pdx)(dq - Fdx - \frac{1}{3}F_q(dp - qdx) + K(dy - pdx)).$$

The normal conformal connection of this geometry is given by

$$(38) \quad \hat{\omega}^c = \begin{pmatrix} \Omega_3 & -\frac{1}{2}\Omega_6 & -\Omega_4 & -\Omega_5 & 0 \\ \theta^1 & \Omega_3 - \Omega_1 & -\theta^4 & 0 & -\Omega_5 \\ \theta^2 & -\Omega_2 & 0 & -\theta^4 & \Omega_4 \\ \theta^3 & 0 & -\Omega_2 & \Omega_1 - \Omega_3 & -\frac{1}{2}\Omega_6 \\ 0 & \theta^3 & -\theta^2 & \theta^1 & -\Omega^3 \end{pmatrix}.$$

*Proof.* Let us define on  $P^c$  the symmetric two-contravariant tensor field

$$\hat{g} = (\theta^2)^2 - 2\theta^1\theta^3$$

of signature  $(++-0000000)$ . The degenerate directions of  $\hat{g}$  are precisely those tangent to the fibres of  $P^c \rightarrow \mathcal{S}$

$$\hat{g}(X_i, \cdot) = 0, \quad \text{for } i = 4, 5, 6, 7, 8, 9, 10.$$

The Lie derivatives of  $\hat{g}$  along the degenerate directions are as follows

$$(39) \quad L_{X_4}\hat{g} = \frac{u_3^3}{u_1^3} W(\theta^1)^2, \quad L_{X_7}\hat{g} = 2\hat{g},$$

and

$$(40) \quad L_{X_i}\hat{g} = 0 \quad \text{for } i = 5, 6, 8, 9, 10.$$

Thus, if only  $W = 0$ , all the degenerate directions but  $X_7$  are isometries of  $\hat{g}$  whereas  $X_7$  is a conformal symmetry. This allows us to *project*  $\hat{g}$  to a Lorentzian conformal metric  $[g]$  on the solution space  $\mathcal{S}$ . By construction the conformal geometries of

contact equivalent ODEs are equivalent. By construction the conformal geometries of contact equivalent ODEs are equivalent.

To prove that  $\widehat{\omega}^c$  is the normal connection we notice that the condition  $W = 0$  means  $\mathbf{A}_1^c = 0$  which causes  $\mathbf{A}_2^c = \dots = \mathbf{A}_8^c = 0$ . Thus, structural equations (34) do not contain the non-constant terms proportional to  $\theta^4$  and the curvature  $\widehat{K}^c$  is horizontal over  $\mathcal{S}$ . As a consequence,  $\widehat{\omega}^c$  is a connection over  $\mathcal{S}$ . To assure that it is normal we rearrange  $\widehat{\omega}^c$  according to the five-dimensional representation (35) and we check the normality conditions.  $\square$

**Remark 3.5.** In order to find the explicit expression for  $g$  in a coordinate system  $(c_1, c_2, c_3)$  on  $\mathcal{S}$  we would have to find the general solution  $y = f(x; c_1, c_2, c_3)$  of the underlying ODE, then solve the system  $(y = f, p = f_x, q = f_{xx})$  with respect to  $c_1, c_2, c_3$  and substitute these formulae into (37).

The conformal grading of  $\mathfrak{o}(3, 2)$  is as follows

$$(41) \quad \mathfrak{o}(3, 2) = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1,$$

where

$$\mathfrak{g}_{-1} = \langle e_1, e_2, e_3 \rangle, \quad \mathfrak{g}_0 = \langle e_4, e_5, e_6, e_7 \rangle, \quad \mathfrak{g}_1 = \langle e_8, e_9, e_{10} \rangle$$

and the base  $(e_i)$  is given by the decomposition of (38):

$$\widehat{\omega}^c = \theta^1 e_1 + \theta^2 e_2 + \theta^3 e_3 + \theta^4 e_4 + \Omega_1 e_5 + \Omega_2 e_6 + \Omega_3 e_7 + \Omega_4 e_8 + \Omega_5 e_9 + \Omega_6 e_{10}.$$

The curvature is equal to

$$\widehat{K}^c = \begin{pmatrix} 0 & DP_1 & DP_2 & DP_3 & 0 \\ 0 & 0 & 0 & 0 & DP_3 \\ 0 & 0 & 0 & 0 & -DP_2 \\ 0 & 0 & 0 & 0 & DP_1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

with the following components of the Cotton tensor on  $P^c$

$$\begin{aligned} DP_1 &= -\frac{1}{2}\mathbf{C}_1^c \theta^1 \wedge \theta^2 - \mathbf{B}_2^c \theta^1 \wedge \theta^3 - \mathbf{B}_3^c \theta^2 \wedge \theta^3, \\ DP_2 &= -\mathbf{B}_2^c \theta^1 \wedge \theta^2 - 2\mathbf{B}_3^c \theta^1 \wedge \theta^3 - \mathbf{B}_4^c \theta^2 \wedge \theta^3, \\ DP_3 &= -\mathbf{B}_3^c \theta^1 \wedge \theta^2 - \mathbf{B}_4^c \theta^1 \wedge \theta^3 - \mathbf{B}_1^c \theta^2 \wedge \theta^3. \end{aligned}$$

Finally, we pull-back these formula to  $J^2$  through  $u_1 = u_3 = 1, u_2 = u_4 = u_5 = u_6 = 0$  and get

$$\begin{aligned} DP_1 &= (\frac{1}{2}M_p + \frac{1}{6}F_q M_q + \frac{1}{6}F_{qq} K_y + K_q L_q - \frac{1}{6}K^2 F_{qqq} + \\ &\quad + \frac{1}{6}K_q F_{qqy} - \frac{1}{6}F_{qqyy} - \frac{1}{3}F_{qqq} K_q K + \frac{1}{3}F_{qqy} K) \omega^1 \wedge \omega^2 \\ &\quad + \frac{1}{2}(M_q - K_{qq} K - 2K_{qq} K_q + K_{qqy}) \omega^1 \wedge \widetilde{\omega}^3 + \\ &\quad - \frac{1}{2}L_{qq} \omega^2 \wedge \widetilde{\omega}^3, \\ DP_2 &= \frac{1}{2}(M_q - K_{qq} K - 2K_{qq} K_q + K_{qqy}) \omega^1 \wedge \omega^2 + \\ &\quad - L_{qq} \omega^1 \wedge \widetilde{\omega}^3 + \frac{1}{2}K_{qq} \omega^2 \wedge \widetilde{\omega}^3, \\ DP_3 &= -\frac{1}{2}L_{qq} \omega^1 \wedge \omega^2 + \frac{1}{2}K_{qq} \omega^1 \wedge \widetilde{\omega}^3 - \frac{1}{6}F_{qqq} \omega^2 \wedge \widetilde{\omega}^3. \end{aligned}$$

The formulae for the conformal connection and curvature (in a slightly different notation) are given in [23].

**Example 3.6.** The simplest nontrivial equations with vanishing Wünschmann invariant and are those with four-dimensional transitive groups of contact symmetries. Up to contact transformations they are as follows: ( $\mu \in \mathbb{R}$ )

$$\begin{aligned} F &= \mu \left( \frac{q^2}{1-p^2} - p^2 + 1 \right)^{3/2} - \frac{3q^2 p}{1-p^2} + p^3 - p^2, \\ F &= \mu \frac{(2qy - p^2)^{3/2}}{y^2}, \\ F &= 4\mu(q - p^2)^{3/2} + 6qp - 4p^3, \\ F &= \mu \left( \frac{q^2}{p^2} + p^2 \right)^{3/2} + 3\frac{q^2}{p} + p^3, \\ F &= (q^2 + 1)^{3/2}, \quad F = q^{3/2}. \end{aligned}$$

First of these examples has the symmetry group  $CO(3)$  and three next have symmetry  $CO(1, 2)$ .

**3.5. Contact projective geometry on first jet space.** The connection  $\hat{\omega}^c$  gives rise to not only the above conformal structure but also a geometry on the first jet space  $J^1$ . As we know, see corollary 3.2, there are two basic contact invariants in the curvature  $\hat{K}^c$ :  $W$  and  $F_{qqqq}$ . The condition  $W = 0$  yields the conformal geometry. Let us now examine the second possibility

$$F_{qqqq} = 0.$$

The above condition yields  $\mathbf{B}_1^c = \mathbf{B}_2^c = \mathbf{B}_3^c = \mathbf{B}_4^c = 0$ , which removes all  $\theta^i \wedge \theta^3$  terms in the curvature and turns  $\hat{\omega}^c$  into a  $\mathfrak{sp}(4, \mathbb{R})$  Cartan connection on  $H_7 \rightarrow P^c \rightarrow J^1$ , since in the curvature there are only terms proportional to  $\theta^1, \theta^2, \theta^4$ , horizontal with respect to  $P^c \rightarrow J^1$ . A natural question is to what geometric structure  $\hat{\omega}^c$  is now related. This geometry is the contact projective structure on  $J^1$  generated by the family of solutions of the ODE.

**3.5.1. Contact projective geometry.** This geometry has been exhaustively analyzed in [17], see also [7, 8]. We will not discuss the general theory here but focus on an application of the three-dimensional case to the ODEs. The definition of contact-projective geometry, see D. Fox [17], adapted to our situation is the following.

**Definition 3.7.** A contact projective structure on the first jet space  $J^1$  is given by the following data.

- i) The contact distribution  $\mathcal{C}$ , that is the distribution annihilated by

$$\omega^1 = dy - p dx.$$

- ii) A family of unparameterized curves everywhere tangent to  $\mathcal{C}$  and such that:
  - a) for a given point and a direction in  $\mathcal{C}$  there is exactly one curve passing through that point and tangent to that direction,
  - b) curves of the family are among unparameterized geodesics for some linear connection on  $J^1$ .

A contact projective structure on  $J^1$  is equivalently given by a family of linear connections, whose geodesic spray contains the family of curves. For  $\nabla$  to belong to this class one needs

$$(42) \quad \nabla_V V = \lambda(V)V$$

along every curve in the family, where  $X$  denotes a tangent field to the considered curve and  $\lambda(X)$  is a function. Given two such connections  $\nabla$  and  $\tilde{\nabla}$ , their difference is a  $(2, 1)$ -type tensor field

$$A(X, Y) = \tilde{\nabla}_X Y - \nabla_X Y,$$

for all  $X$  and  $Y$ . Simultaneously we have  $A(V, V) = \mu(V)V$  for  $V \in \mathcal{C}$ , where  $\mu(V) = \tilde{\lambda}(V) - \lambda(V)$  and  $\mu_w$  at a point  $w \in J^1$  is a covector on the vector space  $\mathcal{C}_w$ . By polarization we obtain

$$(43) \quad A(X, Y) + A(Y, X) = \mu(X)Y + \mu(Y)X, \quad X, Y \in \mathcal{C}.$$

The connections associated to a contact projective structure, when considered at a point, form an affine space characterized by the above  $A$ .

Let us describe  $A$  more explicitly. We choose a frame  $(e_1 = \partial_y, e_2 = \partial_p, e_3 = \partial_x + p\partial_y)$  and denote the dual frame by  $(\sigma^1, \sigma^2, \sigma^3)$ . In particular  $\omega^1 = \sigma^1$  and  $\mathcal{C} = \langle e_2, e_3 \rangle$ . Let  $i, j, \dots = 1, 2, 3$  and  $I, J, \dots = 2, 3$ . Now  $\nabla_j e_i = \Gamma_{ij}^k e_k$ ,  $A_{ij}^k = \tilde{\Gamma}_{ij}^k - \Gamma_{ij}^k$ ,  $\mu = \mu^I e_I$  and (43) reads

$$(44) \quad A_{(IJ)}^k = \mu_I \delta_J^k.$$

Relevant components are equal to  $A_{22}^1 = A_{(23)}^1 = A_{33}^1 = 0$ ,  $A_{22}^2 = \mu_2$ ,  $A_{(23)}^2 = \frac{1}{2}\mu_3$ ,  $A_{33}^2 = 0$ ,  $A_{22}^3 = 0$ ,  $A_{(23)}^3 = \frac{1}{2}\mu_2$ ,  $A_{33}^3 = \mu_3$  and the rest of  $A_{ij}^k$  is free. Elementary calculations assure us that the class of admissible connections is a 20-dimensional subspace of 27-dimensional space of all linear connections on  $J^1$ . Another constraint for the connections is given by

$$(45) \quad (\nabla_V \omega^1)V = \nabla_V(\omega^1(V)) = 0, \quad V \in \mathcal{C}.$$

In our frame this is equivalent to  $\Gamma_{(IJ)}^k = 0$ . Combining eq. (44) and (45) we obtain

**Proposition 3.8.** *The following quantities are invariant with respect to a choice of a connection in the class distinguished by a contact projective structure on  $J^1$*

$$(46a) \quad \Gamma_{22}^1 = 0, \quad \Gamma_{(23)}^1 = 0, \quad \Gamma_{33}^1 = 0,$$

$$(46b) \quad \Gamma_{22}^3, \quad 2\Gamma_{(23)}^3 - \Gamma_{22}^2, \quad \Gamma_{33}^3 - 2\Gamma_{(23)}^2, \quad \Gamma_{33}^2.$$

*The connection coefficients are calculated in a frame  $(e_i)$  such that  $\mathcal{C} = \langle e_2, e_3 \rangle$ .*

*Values of four the unspecified combinations (46b) define a contact projective structure.*

Among the above connections there is a distinguished subclass of those connections which covariantly preserve the distribution  $\mathcal{C}$ . We shall call them compatible connections. They satisfy not only (45) but a stronger condition

$$\nabla_X \omega^1 = \phi(X)\omega^1, \quad \text{for all } X,$$

with some one-form  $\phi$ . Since  $\omega^1$  is non-closed a compatible connection has nonvanishing torsion.

**3.5.2. Contact projective geometries from ODEs.** It is obvious that the family of solutions of an arbitrary third-order ODE satisfies the conditions i) and ii a) of definition 3.7. (Condition ii a) is satisfied with the possible exception of the direction  $\partial_p$ , which belongs to  $\mathcal{C}$  but it is not tangent to any solution in general. However, this exception is irrelevant since our consideration is local on  $TJ^1$ .) We ask when the solutions form a subfamily of geodesics for a linear connection.

**Lemma 3.9.** *A third-order ODE  $y''' = F(x, y, y', y'')$  defines a contact-projective structure on  $J^1$  if and only if  $F_{qqqq} = 0$ . Moreover, the quantities (46b) are given by*

$$(47) \quad \begin{aligned} \Gamma_{22}^3 &= a_3, & 2\Gamma_{(23)}^3 - \Gamma_{22}^2 &= a_2, \\ \Gamma_{33}^3 - 2\Gamma_{(23)}^2 &= a_1, & \Gamma_{33}^2 &= -a_0, \end{aligned}$$

where

$$F = a_3 q^3 + a_2 q^2 + a_1 q + a_0.$$



*Proof.* The field  $V = \frac{d}{dx}$  tangent to a solution  $(x, f(x), f'(x))$  equals  $V = f''e_2 + e_3$  in the frame  $(e_i)$ . The geodesic equations (42) read

$$\begin{aligned} (f'')^2 \Gamma_{22}^1 + 2f''\Gamma_{(23)}^1 + \Gamma_{33}^1 &= 0, \\ f''' + (f'')^2 \Gamma_{22}^2 + 2f''\Gamma_{(23)}^2 + \Gamma_{33}^2 &= \lambda(V)f'', \\ (f'')^2 \Gamma_{22}^3 + 2f''\Gamma_{(23)}^3 + \Gamma_{33}^3 &= \lambda(V). \end{aligned}$$

First of these equations is equivalent to (46a). From the remaining equations we have that

$$f''' = \Gamma_{22}^3 f''^3 + (2\Gamma_{(23)}^3 - \Gamma_{22}^2) f''^2 + (\Gamma_{33}^3 - 2\Gamma_{(23)}^2) f'' - \Gamma_{33}^2,$$

is satisfied along every solution.  $\square$

The algebra  $\mathfrak{sp}(4, \mathbb{R}) \cong \mathfrak{o}(3, 2)$  has the following grading (apart from those of (33) and (41))

$$(48) \quad \mathfrak{sp}(4, \mathbb{R}) = \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \mathfrak{g}_2,$$

which reads in the base (32):

$$\begin{aligned} \mathfrak{g}_{-2} &= \langle e_1 \rangle, & \mathfrak{g}_{-1} &= \langle e_2, e_4 \rangle, \\ \mathfrak{g}_0 &= \langle e_3, e_5, e_7, e_9 \rangle, \\ \mathfrak{g}_1 &= \langle e_6, e_8 \rangle, & \mathfrak{g}_2 &= \langle e_{10} \rangle. \end{aligned}$$

After calculating Tanaka's normality conditions by the method of section 3.2.1, we observe that  $\hat{\omega}^c$  is the normal connection with respect to the grading (48). In this manner we have re-proved a known fact that to a three-dimensional contact projective geometry there is associated the unique normal  $\mathfrak{sp}(4, \mathbb{R})$ -valued Cartan connection.

**Proposition 3.10.** *If the contact projective geometry on  $J^1$  exists, then  $\hat{\omega}^c$  of theorem 3.1 is the normal Cartan connection for this geometry.*

From  $\hat{\omega}^c$  one may reconstruct the compatible connections. To do this we just observe that first, second and fourth equation of (34) can be written as

$$\underbrace{\begin{pmatrix} d\theta^1 \\ d\theta^2 \\ d\theta^4 \end{pmatrix} + \begin{pmatrix} -\Omega_1 & 0 & 0 \\ -\Omega_2 & -\Omega_3 & \theta^3 \\ -\Omega_4 & -\Omega_5 & \Omega_3 - \Omega_1 \end{pmatrix}}_{\hat{\Gamma}} \wedge \begin{pmatrix} \theta^1 \\ \theta^2 \\ \theta^4 \end{pmatrix} = \begin{pmatrix} \theta^4 \wedge \theta^2 \\ \theta^4 \wedge \theta^3 \\ 0 \end{pmatrix}.$$

The three by three matrix denoted by  $\hat{\Gamma}$  is the  $\mathfrak{g}_0 \oplus \mathfrak{g}_1$ -part of  $\hat{\omega}^c$ . The following proposition holds.

**Proposition 3.11.** *For any section  $s: J^1 \rightarrow P^c$  the pull-back  $s^*\hat{\Gamma}$  written in the coframe  $(s^*\theta^1, s^*\theta^2, s^*\theta^4)$  is a connection compatible with the contact projective geometry.*

*Proof.* First we choose the section  $s_0: J^1 \rightarrow P^c$  given by  $q = 0$ ,  $u_1 = 1$ ,  $u_3 = 1$  and  $u_2 = u_4 = u_5 = u_6 = 0$ . We denote  $\Gamma = s_0^*\hat{\Gamma}$ . In the coframe  $\sigma^1 = s_0^*\theta^1$ ,  $\sigma^2 = s_0^*\theta^2$  and  $\sigma^3 = s_0^*\theta^4$  we have  $-s_0^*\Omega_3 = \Gamma_{22}^2 = \Gamma_{2k}^2 \sigma^k$ ,  $s_0^*\theta^3 = \Gamma_{33}^2 = \Gamma_{3k}^2 \sigma^k$  and so on. Equations (47) follow from (18) and (19), provided that  $F_{qqq} = 0$ .

Next we consider an arbitrary section  $s: J^1 \rightarrow P^c$ . In the local trivialization  $P^c \cong H_7 \times J^1$  we have  $P^c \ni w = (v, x)$ , where  $v \in H_7$ ,  $x \in J^1$  and  $s$  is given by  $x \mapsto (v(x), x)$ . Now  $s^*\hat{\omega}^c(x) = v^{-1}(x)s_0^*\hat{\omega}^c(x)v(x) + v^{-1}(x)dv(x)$ , and  $s^*\hat{\Gamma}$  is the  $\mathfrak{g}_0 \oplus \mathfrak{g}_1$  part of  $s^*\hat{\omega}^c$ . Since the Lie algebra of  $H_7$  is  $\mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \mathfrak{g}_2$ , every  $v(x)$  in the connected component of the identity may be written in the form  $v(x) =$

$v_2(x)v_1(x) = \exp(t_2(x)A_2(x))\exp(t_1(x)A_1(x))$  with  $A_2(x) \in \mathfrak{g}_2$  and  $A_1(x) \in \mathfrak{g}_0 \oplus \mathfrak{g}_1$ . It follows that

$$s^*\hat{\omega}^c(x) = v^{-1}(x)_2 \{v_1^{-1}(x)s_0^*\hat{\omega}^c(x)v_1(x) + v_1^{-1}(x)dv_1(x)\} v_2(x) + v_2^{-1}(x)dv_2(x)$$

But the  $\mathfrak{g}_0 \oplus \mathfrak{g}_1$  part of the quantity in the curly brackets is the connection  $\Gamma = s_0^*\hat{\Gamma}$  written in the coframe  $(s^*\theta^1, s^*\theta^2, s^*\theta^4)$  and  $\text{ad } v^{-1}(x)$  transforms it into other compatible connection, according to (44).  $\square$

**3.6. Six-dimensional conformal geometry in the split signature.** Until now we have not proposed any geometric structure, apart from  $\hat{\omega}^c$ , that could be associated with an ODE of generic type. Motivated by S.-S. Chern's construction we would like to build some kind of conformal geometry starting from an arbitrary ODE, which does not necessarily satisfy the Wünschmann condition. We following theorem holds.

**Theorem 3.12.** *A third order ODE defines a six-dimensional manifold  $M^6$  as space of integral leaves of distribution  $\langle X_8, X_9, X_{10}, X_5 + X_7 \rangle$ . The manifold  $M^6$  is equipped with a split signature conformal geometry, whose normal conformal connection has special holonomy  $\mathfrak{sp}(4, \mathbb{R}) \oplus \mathbb{R}^5$ . The  $\mathfrak{sp}(4, \mathbb{R})$  part of this connection is given by  $\hat{\omega}^c$ .*

The rest of the section is devoted to prove the theorem. Let us define the 'inverse' of the symmetric tensor field  $\hat{g} = 2\theta^1\theta^3 - (\theta^2)^2$  of section 3.4 to be  $\hat{g}_{inv} = \hat{g}^{ij}X_i \otimes X_j = 2X_1X_3 - (X_2)^2$ . We take the  $\mathfrak{o}(2, 1)$ -part of the connection  $\hat{\omega}^c$

$$\Gamma = \begin{pmatrix} \Omega_3 - \Omega_1 & -\theta^4 & 0 \\ -\Omega_2 & 0 & -\theta^4 \\ 0 & -\Omega_2 & \Omega_1 - \Omega_3 \end{pmatrix},$$

and the Levi-Civita symbol  $\epsilon_{ijk}$  in three dimensions. Next we define a new bilinear form  $\hat{\mathbf{g}}$  on  $P^c$

$$\hat{\mathbf{g}} = \epsilon_{ijk} \hat{g}^{kl} \theta^i \Gamma_L^j.$$

The above method of obtaining  $\hat{\mathbf{g}}$  of the split degenerate signature from  $\hat{g}$  is called the Sparling procedure [38]. The new metric reads

$$(49) \quad \hat{\mathbf{g}} = 2(\Omega_1 - \Omega_3)\theta^2 - 2\Omega_2\theta^1 + 2\theta^4\theta^3$$

and was given in [38] in a slightly different context for the first time. We easily find that its degenerated directions  $X_8, X_9, X_{10}$ , and  $X_5 + X_7$  form an integrable distribution, so that one can consider the six-dimensional space  $M^6$  of its integral leaves. The degenerated directions  $X_8, X_9$ , and  $X_{10}$  are isometries

$$L_{X_6}\hat{\mathbf{g}} = L_{X_8}\hat{\mathbf{g}} = L_{X_9}\hat{\mathbf{g}} = 0,$$

whereas the fourth direction,  $X_5 + X_7$ , is a conformal transformation

$$L_{(X_5+X_7)}\hat{\mathbf{g}} = \hat{\mathbf{g}}.$$

This allows us to project  $\hat{\mathbf{g}}$  to the split signature conformal metric  $[\mathbf{g}]$  on  $M^6$  without any assumptions about the underlying ODE.

It is interesting to study the normal conformal connection associated to this geometry. Since  $P^c$  is a subbundle of the conformal bundle over  $M^6$ , we can calculate the  $\mathfrak{o}(4, 4)$ -valued normal conformal connection (36) at once on  $P^c$ . It

is as follows.

$$\widehat{\mathbf{w}} = \begin{pmatrix} \frac{1}{2}\Omega_1 & 0 & 0 & \frac{1}{2}\Omega_2 & -\frac{1}{2}\Omega_4 & -\frac{1}{2}\Omega_6 & 0 & 0 \\ \Omega_1 - \Omega_3 & \Omega_3 - \frac{1}{2}\Omega_1 & \frac{1}{2}\theta_4 & \frac{1}{2}\Omega_2 & 0 & -w_5^3 & \Omega_5 & -\frac{1}{2}\Omega_4 \\ -\Omega_2 & \Omega_2 & \frac{1}{2}\Omega_1 & w_4^3 & w_5^3 & 0 & \Omega_4 & -\frac{1}{2}\Omega_6 \\ \theta_4 & 0 & 0 & \Omega_3 - \frac{1}{2}\Omega_1 & -\Omega_5 & -\Omega_4 & 0 & 0 \\ \theta_2 & 0 & 0 & \theta_3 & \frac{1}{2}\Omega_1 - \Omega_3 & -\Omega_2 & 0 & 0 \\ \theta_1 & 0 & 0 & \frac{1}{2}\theta_2 & -\frac{1}{2}\theta_4 & -\frac{1}{2}\Omega_1 & 0 & 0 \\ \theta_3 & -\theta_3 & -\frac{1}{2}\theta_2 & 0 & -\frac{1}{2}\Omega_2 & -w_4^3 & \frac{1}{2}\Omega_1 - \Omega_3 & \frac{1}{2}\Omega_2 \\ 0 & \theta_2 & \theta_1 & \theta_3 & \Omega_1 - \Omega_3 & -\Omega_2 & \theta_4 & -\frac{1}{2}\Omega_1 \end{pmatrix},$$

where

$$\begin{aligned} w_4^3 &= \mathbf{A}_4^c \theta^1 + \mathbf{A}_2^c \theta^2 + \mathbf{A}_1^c \theta^4, \\ w_5^3 &= \frac{1}{2}\Omega_6 + \mathbf{A}_3^c \theta^1 + \mathbf{A}_5^c \theta^2 + \mathbf{A}_2^c \theta^4. \end{aligned}$$

It appears that this connection is of very special form. We show that the algebra of its holonomy group is reduced to  $\mathfrak{sp}(4, \mathbb{R}) \oplus \mathbb{R}^5$ . Let us write down the connection as

$$\begin{aligned} \widehat{\mathbf{w}} &= (\Omega_1 - \Omega_3)e_1 - \Omega_2 e_2 + \theta^4 e_3 + \theta^2 e_4 + \theta^1 e_5 + \theta^3 e_6 + \\ &\quad + \Omega_1 e_7 + \Omega_4 e_8 + \Omega_5 e_9 + \Omega_6 e_{10} + w_5^3 e_{11} + w_4^3 e_{12}, \end{aligned}$$

where  $e_1, \dots, e_{12}$  are appropriate matrices in  $\mathfrak{o}(4, 4)$ . The space

$$V = \langle e_1, \dots, e_{12} \rangle \subset \mathfrak{o}(4, 4)$$

is not closed under the commutation relations, however, if we extend  $V$  so that it contains three commutators  $e_{13} = [e_3, e_{12}]$ ,  $e_{14} = [e_5, e_{10}]$  and  $e_{15} = [e_5, e_{12}]$  then  $\langle e_1, \dots, e_{15} \rangle$  is a Lie algebra, a certain semidirect sum of  $\mathfrak{sp}(4, \mathbb{R})$  and  $\mathbb{R}^5$ . Bases of the factors are the following:

$$\mathbb{R}^5 = \langle e_1 + 2e_7 - 2e_{14}, e_{11}, e_{12}, e_{13}, e_{15} \rangle,$$

$$\mathfrak{sp}(4, \mathbb{R}) = \langle e_2 + e_{13}, e_3, e_4, e_5, e_6 - e_{15}, e_7, e_8, e_9, e_{10}, e_{14} \rangle.$$

The matrix of  $\widehat{\mathbf{w}}$  can be transformed to the following conjugated representation, which reveals its structure well

$$\begin{pmatrix} \frac{1}{2}\Omega_1 & \frac{1}{2}\Omega_2 & -\frac{1}{2}\Omega_4 & -\frac{1}{4}\Omega_6 & 2\Omega_2 & -2\mathbf{w}_4^3 & 2\mathbf{w}_5^3 & 0 \\ \theta^4 & \Omega_3 - \frac{1}{2}\Omega_1 & -\Omega_5 & -\frac{1}{2}\Omega_4 & 4\Omega_3 - 4\Omega_1 & -2\Omega_2 & 0 & -2\mathbf{w}_5^3 \\ \theta^2 & \theta^3 & \frac{1}{2}\Omega_1 - \Omega_3 & -\frac{1}{2}\Omega_2 & 4\theta^3 & 0 & 2\Omega_2 & 2\mathbf{w}_4^3 \\ 2\theta^1 & \theta^2 & -\theta^4 & -\frac{1}{2}\Omega_1 & 0 & -4\theta^3 & 4\Omega_1 - 4\Omega_3 & -2\Omega_2 \\ 0 & 0 & 0 & 0 & \frac{1}{2}\Omega_1 & \frac{1}{2}\Omega_2 & \frac{1}{2}\Omega_4 & \frac{1}{4}\Omega_6 \\ 0 & 0 & 0 & 0 & \theta^4 & \Omega_3 - \frac{1}{2}\Omega_1 & \Omega_5 & \frac{1}{2}\Omega_4 \\ 0 & 0 & 0 & 0 & -\theta^2 & -\theta^3 & \frac{1}{2}\Omega_1 - \Omega_3 & -\frac{1}{2}\Omega_2 \\ 0 & 0 & 0 & 0 & -2\theta^1 & -\theta^2 & -\theta^4 & -\frac{1}{2}\Omega_1 \end{pmatrix}.$$

$\widehat{\mathbf{w}}$  has the following block structure in this representation

$$\widehat{\mathbf{w}} = \begin{pmatrix} \widehat{\omega}^c & \widehat{\tau} \\ 0 & -\sigma \widehat{\omega}^c \sigma \end{pmatrix},$$

where

$$\sigma = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}.$$

Surprisingly enough the  $\mathfrak{sp}(4, \mathbb{R})$ -part of  $\widehat{\mathbf{w}}$ , given by the diagonal blocks, is totally determined by the  $\mathfrak{sp}(4, \mathbb{R})$  connection  $\widehat{\omega}^c$ . In particular, this relation holds when  $W = 0$  and  $\widehat{\omega}^c$  is a conformal connection itself. In this case we have rather an unexpected link between conformal connections in dimensions three and six.

**3.7. Further reduction and geometry on five-dimensional bundle.** Theorem 3.1 is a starting point for further reduction of the structural group since one can use the non-constant invariants in (34) to eliminate more variables  $u_\mu$ . From this point of view third-order ODEs fall into three main classes:

- i)  $W = 0$ ,  $F_{qqqq} = 0$ . This class contains the equations equivalent to  $y''' = 0$  and is fully characterized by the corollary 3.3.
- ii)  $W = 0$ ,  $F_{qqqq} \neq 0$ . It is not interesting from the geometric point of view since it does not contain equations with five-dimensional or larger symmetry groups. One may prove this by doing full group reduction, see [26].
- iii)  $W \neq 0$ . This class leads to a Cartan connection on a five-dimensional bundle and is studied below.

Let us assume that  $W \neq 0$  and continue reduction by setting  $\mathbf{A}_1^c = 1$ ,  $\mathbf{A}_2^c = 0$ , which gives

$$u_1 = \sqrt[3]{W} u_3, \quad u_5 = \frac{1}{3} \frac{W_q}{\sqrt[3]{W^2}}.$$

At this moment the auxiliary variable  $u_6$  which was introduced by the prolongation becomes irrelevant and may be set equal to zero

$$u_6 = 0.$$

In second step we choose

$$u_2 = \frac{1}{3}Zu_3$$

and finally

$$u_4 = \frac{1}{9} \frac{W_q}{\sqrt[3]{W^2}} M - \frac{1}{3} \sqrt[3]{W} Z_q.$$

The coframe and the underlying bundle  $P^c$  of theorem 3.1 have been reduced to dimension five according to the following

**Theorem 3.13** (S.-S. Chern). *A third-order ODE  $y''' = F(x, y, y', y'')$  satisfying the contact invariant condition  $W \neq 0$  and considered modulo contact transformations of variables, uniquely defines a 5-dimensional bundle  $P_5^c$  over  $J^2$  and an invariant coframe  $(\theta^1, \dots, \theta^4, \Omega)$  on it. In local coordinates  $(x, y, p, q, u)$  this coframe is given by*

$$\begin{aligned} \theta^1 &= \sqrt[3]{W} u \omega^1, \\ \theta^2 &= \frac{1}{3} Z u \omega^1 + u \omega^2, \\ \theta^3 &= \frac{u}{\sqrt[3]{W}} \left( K + \frac{1}{18} Z^2 \right) \omega^1 + \frac{u}{3\sqrt[3]{W}} (Z - F_q) \omega^2 + \frac{u}{\sqrt[3]{W}} \omega^3, \\ (50) \quad \theta^4 &= \left( \frac{1}{9} \frac{W_q}{\sqrt[3]{W^2}} Z - \frac{1}{3} \sqrt[3]{W} Z_q \right) \omega^1 + \frac{W_q}{3\sqrt[3]{W^2}} \omega^2 + \sqrt[3]{W} \omega^4, \\ \Omega &= \left( \left( \frac{1}{9} W_q \mathcal{D}Z - \frac{1}{27} W_q Z^2 + \frac{1}{9} W_p Z \right) \frac{1}{W} - \frac{1}{3} Z_p - \frac{1}{9} F_q Z_q \right) \omega^1 \\ &\quad + \left( \frac{W_p}{3W} - \frac{1}{3} Z_q \right) \omega^2 + \frac{W_q}{3W} \omega^3 + \frac{1}{3} F_q \omega^4 + \frac{du}{u}. \end{aligned}$$

where  $\omega^i$  are defined by the ODE via (10). The exterior derivatives of these forms read

$$\begin{aligned} d\theta^1 &= \Omega \wedge \theta^1 - \theta^2 \wedge \theta^4, \\ d\theta^2 &= \Omega \wedge \theta^2 + \mathbf{a}^c \theta^1 \wedge \theta^4 - \theta^3 \wedge \theta^4, \\ (51) \quad d\theta^3 &= \Omega \wedge \theta^3 + \mathbf{b}^c \theta^1 \wedge \theta^2 + \mathbf{c}^c \theta^1 \wedge \theta^3 - \theta^1 \wedge \theta^4 + \mathbf{e}^c \theta^2 \wedge \theta^3 + \mathbf{a}^c \theta^2 \wedge \theta^4, \\ d\theta^4 &= \mathbf{f}^c \theta^1 \wedge \theta^2 + \mathbf{g}^c \theta^1 \wedge \theta^3 + \mathbf{h}^c \theta^1 \wedge \theta^4 + \mathbf{k}^c \theta^2 \wedge \theta^3 - \mathbf{e}^c \theta^2 \wedge \theta^4, \\ d\Omega &= \mathbf{l}^c \theta^1 \wedge \theta^2 + (\mathbf{f}^c - \mathbf{a}^c \mathbf{k}^c) \theta^1 \wedge \theta^3 + \mathbf{m}^c \theta^1 \wedge \theta^4 + \mathbf{g}^c \theta^2 \wedge \theta^3 + \mathbf{h}^c \theta^2 \wedge \theta^4. \end{aligned}$$

The basic invariants for (51) (i.e. generating the full set of invariants by consecutive taking of coframe derivatives) are  $\mathbf{a}^c, \mathbf{b}^c, \mathbf{e}^c, \mathbf{h}^c, \mathbf{k}^c$ :

$$\begin{aligned} \mathbf{a}^c &= \frac{1}{\sqrt[3]{W^2}} \left( K + \frac{1}{18} Z^2 + \frac{1}{9} Z F_q - \frac{1}{3} \mathcal{D}Z \right), \\ \mathbf{b}^c &= \frac{1}{3u\sqrt[3]{W^2}} \left( \frac{1}{27} F_{qq} Z^2 + \left( K_q - \frac{1}{3} Z_p - \frac{2}{9} F_q Z_q \right) Z + \right. \\ &\quad \left. + \left( \frac{1}{3} \mathcal{D}Z - 2K \right) Z_q + Z_y + F_{qq} K - 3K_p - K_q F_q - F_{qy} + W_q \right), \\ (52) \quad \mathbf{e}^c &= \frac{1}{u} \left( \frac{1}{3} F_{qq} + \frac{1}{W} \left( \frac{2}{9} W_q Z - \frac{2}{3} W_p - \frac{2}{9} W_q F_q \right) \right), \\ \mathbf{h}^c &= \frac{1}{3u\sqrt[3]{W}} \left( \left( \frac{1}{9} W_q Z^2 - \frac{1}{3} W_p Z + W_y - \frac{1}{3} W_q \mathcal{D}Z \right) \frac{1}{W} + \right. \\ &\quad \left. + \mathcal{D}Z_q + \frac{1}{3} F_q Z_q \right), \end{aligned}$$

$$\mathbf{k}^c = \frac{1}{u^2 \sqrt[3]{W}} \left( \frac{2W_q^2}{9W} - \frac{W_{qq}}{3} \right).$$

Our next aim is to obtain a Cartan connection. First of all we study the most symmetric case to find the Lie algebra of a connection. We assume that all the functions  $\mathbf{a}^c, \dots, \mathbf{m}^c$  are constant. Having applied the exterior derivative to (51) we get that  $\mathbf{b}^c, \dots, \mathbf{m}^c$  are equal to zero and  $\mathbf{a}^c$  is an arbitrary real constant  $\mu$ . In this case the equations (51) become the Maurer-Cartan equations for the algebra  $\mathbb{R}^2 \oplus_\mu \mathbb{R}^3$ . Straightforward calculations show that this case corresponds to a general linear equation with constant coefficients.

**Corollary 3.14.** *A third-order ODE is contact equivalent to*

$$y''' = -2\mu y' + y,$$

where  $\mu$  is an arbitrary constant if and only if it satisfies

$$(53) \quad \begin{aligned} 1) \quad & W \neq 0, \\ 2) \quad & \frac{1}{\sqrt[3]{W^2}} \left( K + \frac{1}{18} Z^2 + \frac{1}{9} Z F_q - \frac{1}{3} \mathcal{D}Z \right) = \mu \\ 3) \quad & 2W_q^2 - 3W_{qq}W = 0. \end{aligned}$$

Such an equation has the five-dimensional algebra  $\mathbb{R}^2 \oplus_\mu \mathbb{R}^3$  of infinitesimal contact symmetries. The equations with different constants  $\mu_1$  and  $\mu_2$  are non-equivalent.

*Proof.* Assume that  $\mathbf{a}^c = \mu$ ,  $\mathbf{k}^c = 0$ . It follows from  $d^2\theta^i = 0$  and  $d^2\Omega = 0$ , that this assumption makes other functions in (51) vanish. Put  $y''' = -2\mu y' + y$  into the formulae of theorem 3.13 and check that it satisfies  $\mathbf{a}^c = \mu$ ,  $\mathbf{k}^c = 0$ . Every equation satisfying  $\mathbf{a}^c = \mu$ ,  $\mathbf{k}^c = 0$  is contact equivalent to it by virtue of the Cartan equivalence method.  $\square$

Now we immediately find a family of Cartan connections

**Theorem 3.15.** *An ODE which satisfies the condition*

$$\frac{1}{\sqrt[3]{W^2}} \left( K + \frac{1}{18} Z^2 + \frac{1}{9} Z F_q - \frac{1}{3} \mathcal{D}Z \right) = \mu$$

has the solution space equipped with the following  $\mathbb{R}^2$ -valued linear torsion-free connection

$$\widehat{\omega}_\mu = \begin{pmatrix} -\Omega & -\theta^4 & 0 \\ \mu\theta^4 & -\Omega & -\theta^4 \\ -\theta^4 & \mu\theta^4 & -\Omega \end{pmatrix}.$$

Its curvature reads

$$\begin{pmatrix} R_1^1 & R_2^1 & 0 \\ -\mu R_2^1 & R_1^1 & R_2^1 \\ R_2^1 & -\mu R_2^1 & R_1^1 \end{pmatrix}$$

with

$$\begin{aligned} R_1^1 &= (\mu \mathbf{g}^c + \mathbf{k}^c) \theta^1 \wedge \theta^2 + (\mu \mathbf{k}^c - \mathbf{g}^c) \theta^1 \wedge \theta^3 - \mathbf{g}^c \theta^2 \wedge \theta^3, \\ R_2^1 &= -\mathbf{f}^c \theta^1 \wedge \theta^2 - \mathbf{g}^c \theta^1 \wedge \theta^3 - \mathbf{k}^c \theta^2 \wedge \theta^3. \end{aligned}$$

The connection is flat if and only if the related ODE is contact equivalent to  $y''' = -2\mu y' + y$

*Proof.* The condition  $\mathbf{a}^c = \mu$  together with its differential consequences  $\mathbf{b}^c = \mathbf{c}^c = \mathbf{e}^c = \mathbf{h}^c = \mathbf{m}^c = 0$  and  $\mathbf{l}^c = -\mathbf{k}^c - \mu \mathbf{g}^c$  is the necessary and sufficient condition for the curvature of  $\widehat{\omega}_\mu$  to be horizontal.  $\square$

It follows that every ODE as above has its space of solution equipped with a geometric structure consisting of

- i) Reduction of  $\mathfrak{gl}(3, \mathbb{R})$  to  $\mathbb{R}^2$  represented by

$$\begin{pmatrix} a_1 & a_2 & 0 \\ -\mu a_2 & a_1 & a_2 \\ a_2 & -\mu a_2 & a_1 \end{pmatrix}.$$

- ii) A linear torsion-free connection  $\Gamma$  taking values in this  $\mathbb{R}^2$ .

The structure is an example of a geometry with special holonomy. The algebra  $\mathbb{R}^2$  is spanned by the unit matrix and

$$m(\mu) = \begin{pmatrix} 0 & 1 & 0 \\ -\mu & 0 & 1 \\ 1 & -\mu & 0 \end{pmatrix},$$

whose action on  $\mathcal{S}$  is more complicated. Its eigenvalue equation

$$\det(u\mathbf{1} - m(\mu)) = u^3 + 2\mu u - 1$$

is the characteristic polynomial of the linear ODE  $y''' = -2\mu y' + y$ . If  $\mu < \frac{3}{4}\sqrt[3]{2}$  the polynomial has three distinct roots and  $m$  is a generator of non-isotropic dilatations acting along the eigenspaces. If  $\mu = \frac{3}{4}\sqrt[3]{2}$  the characteristic polynomial has two roots, one of them double, for other  $\mu$ s there is one eigenvalue. The action is diagonalizable only in the case of three distinct eigenvalues.

#### 4. GEOMETRIES OF ODES MODULO POINT TRANSFORMATIONS OF VARIABLES

**4.1. Cartan connection on seven-dimensional bundle.** Following the scheme of reduction given in section 3 we construct Cartan connection for ODEs modulo point transformations.

**Theorem 4.1** (E. Cartan). *To every third order ODE  $y''' = F(x, y, y', y'')$  there are associated the following data.*

- i) *The principal fibre bundle  $H_3 \rightarrow P^p \rightarrow J^2$ , where  $\dim P^p = 7$ , and  $H_3$  is the three-dimensional group*

$$(54) \quad H_3 = \begin{pmatrix} \sqrt{u_1}, & \frac{1}{2} \frac{u_2}{\sqrt{u_1}}, & 0 & 0 \\ 0 & \frac{u_3}{\sqrt{u_1}}, & 0 & 0 \\ 0 & 0 & \frac{\sqrt{u_1}}{u_3}, & -\frac{1}{2} \frac{u_2}{\sqrt{u_1} u_3} \\ 0 & 0 & 0 & \frac{1}{\sqrt{u_1}} \end{pmatrix}.$$

- ii) *The coframe  $(\theta^1, \theta^2, \theta^3, \theta^4, \Omega_1, \Omega_2, \Omega_3)$ , which defines the  $\mathfrak{co}(2, 1) \oplus \mathbb{R}^3$ -valued Cartan connection  $\hat{\omega}^p$  on  $P^p$  by*

$$(55) \quad \hat{\omega}^p = \begin{pmatrix} \frac{1}{2}\Omega_1 & \frac{1}{2}\Omega_2 & 0 & 0 \\ \theta^4 & \Omega_3 - \frac{1}{2}\Omega_1 & 0 & 0 \\ \theta^2 & \theta^3 & \frac{1}{2}\Omega_1 - \Omega_3 & -\frac{1}{2}\Omega_2 \\ 2\theta^1 & \theta^2 & -\theta^4 & -\frac{1}{2}\Omega_1 \end{pmatrix}.$$

Two 3rd order ODEs  $y''' = F(x, y, y', y'')$  and  $y''' = \bar{F}(x, y, y', y'')$  are locally point equivalent if and only if their associated Cartan connections are locally diffeomorphic, that is there exists a bundle diffeomorphism  $\Phi: \bar{P}^p \supset \bar{\mathcal{O}} \rightarrow \mathcal{O} \subset P^p$  such that

$$\Phi^* \hat{\omega}^p = \overline{\hat{\omega}^p}.$$

Let  $(x, y, p, q, u_1, u_2, u_3) = (x^i, u_\mu)$  be a locally trivializing coordinate system in  $P^p$ . Then the value of  $\hat{\omega}^p$  at the point  $(x^i, u_\mu)$  in  $P^p$  is given by

$$\hat{\omega}^p(x^i, u_\mu) = u^{-1} \omega^p u + u^{-1} du$$

where  $u$  denotes the matrix (54) and

$$\omega^p = \begin{pmatrix} \frac{1}{2}\Omega_1^0 & \frac{1}{2}\Omega_2^0 & 0 & 0 \\ \tilde{\omega}^4 & \Omega_3^0 - \frac{1}{2}\Omega_1^0 & 0 & 0 \\ \omega^2 & \tilde{\omega}^3 & \frac{1}{2}\Omega_1^0 - \Omega_3^0 & -\frac{1}{2}\Omega_2^0 \\ 2\omega^1 & \omega^2 & -\tilde{\omega}^4 & -\frac{1}{2}\Omega_1^0 \end{pmatrix}$$

is the connection  $\hat{\omega}^p$  calculated at the point  $(x^i, u_1 = 1, u_2 = 0, u_3 = 1)$ . The forms  $\omega^1, \omega^2, \tilde{\omega}^3, \omega^4$  read

$$\begin{aligned} \omega^1 &= dy - p dx, \\ \omega^2 &= dp - q dx, \\ \tilde{\omega}^3 &= dq - F dx - \frac{1}{3} F_q (dp - q dx) + K(dy - p dx), \\ \tilde{\omega}^4 &= dx + \frac{1}{6} F_{qq} (dy - p dx). \end{aligned}$$

The forms  $\Omega_1^0, \dots, \Omega_6^0$  read

$$\begin{aligned} \Omega_1^0 &= -(3K_q + \frac{2}{9} F_{qq} F_q + \frac{2}{3} F_{qp}) \omega^1 + \frac{1}{6} F_{qq} \omega^2, \\ \Omega_2^0 &= (L + \frac{1}{6} F_{qq} K) \omega^1 - (2K_q + \frac{1}{9} F_{qq} F_q + \frac{1}{3} F_{qp}) \omega^2 + \frac{1}{6} F_{qq} \tilde{\omega}^3 - K \tilde{\omega}^4, \\ \Omega_3^0 &= -(2K_q + \frac{1}{6} F_{qq} F_q + \frac{1}{3} F_{qp}) \omega^1 + \frac{1}{3} F_{qq} \omega^2 + \frac{1}{3} F_q \tilde{\omega}^4. \end{aligned}$$

*Proof.* We begin with the  $G_p$ -structure on  $J^2$  (13), which encodes an ODE up to point transformations. In the usual locally trivializing coordinate system  $(x, y, p, q, u_1, \dots, u_8)$  on  $G_p \times J^2$  the fundamental form  $\theta^i$  is given by

$$\begin{aligned} \theta^1 &= u_1 \omega^1, \\ \theta^2 &= u_2 \omega^2 + u_3 \omega^3, \\ \theta^3 &= u_4 \omega^1 + u_5 \omega^2 + u_6 \omega^3, \\ \theta^4 &= u_8 \omega^1 + u_7 \omega^4. \end{aligned}$$

We repeat the procedure of section 3.2. We choose a connection by the minimal torsion requirement and then reduce  $G_p \times J^2$  using the constant torsion property. We differentiate  $\theta^i$  and gather the  $\theta^j \wedge \theta^k$  terms into

$$\begin{aligned} d\theta^1 &= \Omega_1 \wedge \theta^1 + \frac{u_1}{u_3 u_7} \theta^4 \wedge \theta^2, \\ d\theta^2 &= \Omega_2 \wedge \theta^1 + \Omega_3 \wedge \theta^2 + \frac{u_3}{u_6 u_7} \theta^4 \wedge \theta^3, \\ d\theta^3 &= \Omega_4 \wedge \theta^1 + \Omega_5 \wedge \theta^2 + \Omega_6 \wedge \theta^3, \\ d\theta^4 &= \Omega_8 \wedge \theta^1 + \Omega_9 \wedge \theta^2 + \Omega_7 \wedge \theta^4 \end{aligned} \tag{56}$$



with the auxiliary connection forms  $\Omega_\mu$  containing the differentials of  $u_\mu$  and terms proportional to  $\theta^i$ . Then we reduce  $G_p \times J^2$  by

$$u_6 = \frac{u_3^2}{u_1}, \quad u_7 = \frac{u_1}{u_3}.$$

Subsequently, we get formulae identical to (26), (27):

$$u_5 = \frac{u_3}{u_1} \left( u_2 - \frac{1}{3} u_3 F_q \right),$$

$$u_4 = \frac{u_3^2}{u_1} K + \frac{u_2^2}{2u_1}$$

and also

$$u_8 = \frac{u_1}{6u_3} F_{qq}.$$

After these substitutions the structural equations for  $\theta^i$  are the following

$$\begin{aligned} d\theta^1 &= \Omega_1 \wedge \theta^1 + \theta^4 \wedge \theta^2, \\ d\theta^2 &= \Omega_2 \wedge \theta^1 + \Omega_3 \wedge \theta^2 + \theta^4 \wedge \theta^3, \\ d\theta^3 &= \Omega_2 \wedge \theta^2 + (2\Omega_3 - \Omega_1) \wedge \theta^3 + \mathbf{A}_1^p \theta^4 \wedge \theta^1, \\ d\theta^4 &= (\Omega_1 - \Omega_3) \wedge \theta^4 + \mathbf{B}_1^p \theta^2 \wedge \theta^1 + \mathbf{B}_2^p \theta^3 \wedge \theta^1, \end{aligned}$$

with some functions  $\mathbf{A}_1^p, \mathbf{B}_1^p, \mathbf{B}_2^p$ . But now, in contrast to the contact case, the forms  $\Omega_1, \Omega_2, \Omega_3$  are defined by the above equations without any ambiguity, thus there is no need to prolong and we have the rigid coframe on the seven-dimensional bundle  $P^p \rightarrow J^2$ .  $\square$

**4.1.1. Curvature.** Further analysis of the coframe of theorem 4.1 is very similar to what we have done in section 3. The curvature of the connection is given by nonconstant terms in the formulae of exterior differentials of the coframe

$$\begin{aligned} d\theta^1 &= \Omega_1 \wedge \theta^1 + \theta^4 \wedge \theta^2, \\ d\theta^2 &= \Omega_2 \wedge \theta^1 + \Omega_3 \wedge \theta^2 + \theta^4 \wedge \theta^3, \\ d\theta^3 &= \Omega_2 \wedge \theta^2 + (2\Omega_3 - \Omega_1) \wedge \theta^3 + \mathbf{A}_1^p \theta^4 \wedge \theta^1, \\ d\theta^4 &= (\Omega_1 - \Omega_3) \wedge \theta^4 + \mathbf{B}_1^p \theta^2 \wedge \theta^1 + \mathbf{B}_2^p \theta^3 \wedge \theta^1, \\ (57) \quad d\Omega_1 &= -\Omega_2 \wedge \theta^4 + (\mathbf{D}_1^p + 3\mathbf{B}_3^p) \theta^1 \wedge \theta^2 + (3\mathbf{B}_4^p - 2\mathbf{B}_1^p) \theta^1 \wedge \theta^3 \\ &\quad + (2\mathbf{C}_1^p - \mathbf{A}_2^p) \theta^1 \wedge \theta^4 - \mathbf{B}_2^p \theta^2 \wedge \theta^3, \\ d\Omega_2 &= (\Omega_3 - \Omega_1) \wedge \Omega_2 + \mathbf{D}_2^p \theta^1 \wedge \theta^2 + (\mathbf{D}_1^p + \mathbf{B}_3^p) \theta^1 \wedge \theta^3 + \mathbf{A}_3^p \theta^1 \wedge \theta^4 \\ &\quad + (2\mathbf{B}_4^p - \mathbf{B}_1^p) \theta^2 \wedge \theta^3 + \mathbf{C}_1^p \theta^2 \wedge \theta^4, \\ d\Omega_3 &= (\mathbf{D}_1^p + 2\mathbf{B}_3^p) \theta^1 \wedge \theta^2 + 2(\mathbf{B}_4^p - \mathbf{B}_1^p) \theta^1 \wedge \theta^3 + \mathbf{C}_1^p \theta^1 \wedge \theta^4 - 2\mathbf{B}_2^p \theta^2 \wedge \theta^3, \end{aligned}$$

where  $\mathbf{A}_1^p, \mathbf{A}_2^p, \mathbf{A}_3^p, \mathbf{B}_1^p, \mathbf{B}_2^p, \mathbf{B}_3^p, \mathbf{B}_4^p, \mathbf{C}_1^p, \mathbf{D}_1^p, \mathbf{D}_2^p$  are functions on  $P^p$ . We have

**Corollary 4.2.** *The set of basic point relative invariants for third order ODEs is as follows:*

$$\begin{aligned} \mathbf{A}_1^p &= \frac{u_3^3}{u_1^3} W, \\ \mathbf{B}_1^p &= \frac{1}{u_2^2} \left( \frac{1}{18} F_{qqq} F_q + \frac{1}{36} F_{qq}^2 + \frac{1}{6} F_{qqp} \right) - \frac{u_2}{6u_3^3} F_{qqq}, \\ \mathbf{C}_1^p &= \frac{u_3}{u_1^2} \left( 2F_{qq} K + \frac{2}{3} F_q F_{qp} - 2F_{qy} + F_{pp} + 2W_q \right). \end{aligned}$$

All other invariants can be derived from  $\mathbf{A}_1^p$ ,  $\mathbf{B}_1^p$  and  $\mathbf{C}_1^p$  by consecutive differentiation with respect to the frame  $(X_1, X_2, X_3, X_4, X_5, X_6, X_7)$  be the frame dual to  $(\theta^1, \theta^2, \theta^3, \theta^4, \Omega_1, \Omega_2, \Omega_3)$ . Among these derived invariants  $\mathbf{B}_2^p$  and  $\mathbf{B}_4^p$  are important:

$$\mathbf{B}_2^p = \frac{u_1}{6u_3^3} F_{qqq},$$

$$\mathbf{B}_4^p = \frac{1}{u_3^2} \left( K_{qq} + \frac{1}{9} F_{qqq} F_q + \frac{1}{3} F_{qqp} + \frac{1}{12} F_{qq}^2 \right).$$

**Corollary 4.3.** *For a third-order ODE  $y''' = F(x, y, y', y'')$  the following conditions are equivalent.*

- i) *The ODE is point equivalent to  $y''' = 0$ .*
- ii) *It satisfies the conditions  $W = 0$ ,  $F_{qqq} = 0$ ,  $F_{qq}^2 + 6F_{qqp} = 0$  and*

$$2F_{qq}K + \frac{2}{3}F_q F_{qp} - 2F_{qy} + F_{pp} = 0.$$

- iii) *It has the  $\mathfrak{co}(2, 1) \oplus \mathbb{R}^3$  algebra of infinitesimal point symmetries.*

The manifold  $P^p$ , like  $P^c$ , is equipped with the threefold structure of principal bundle over  $J^2$ ,  $J^1$  and  $\mathcal{S}$ .

- $P^p$  is the bundle  $H_3 \rightarrow P^p \rightarrow J^2$  with the fundamental fields  $X_5, X_6, X_7$ .
- It is the bundle  $CO(2, 1) \rightarrow P^p \rightarrow \mathcal{S}$  with the fundamental fields  $X_4, X_5, X_6, X_7$
- It is also the bundle  $H_4 \rightarrow P^p \rightarrow J^1$  with the fundamental fields  $X_3, X_5, X_6, X_7$ .

**4.2. Einstein-Weyl geometry on space of solutions.** We describe in detail the Einstein-Weyl geometry on the solution space.

**4.2.1. Weyl geometry.** A Weyl geometry on  $M^n$  is a pair  $(g, \phi)$  such that  $g$  is a metric of signature  $(k, l)$ ,  $k + l = n$  and  $\phi$  is a one-form and they are given modulo the following transformations

$$\phi \rightarrow \phi + d\lambda, \quad g \rightarrow e^{2\lambda} g.$$

In particular  $[g]$  is a conformal geometry. For any Weyl geometry there exists the Weyl connection; it is the unique torsion-free connection such that

$$\nabla g = 2\phi \otimes g.$$

The Weyl connection takes values in the algebra  $\mathfrak{co}(k, l)$  of  $[g]$ . Let  $(\omega^\mu)$  be a coframe such that for some  $g$  of  $[g]$  is equal  $g = g_{\mu\nu} \omega^\mu \otimes \omega^\nu$  with constant coefficients  $g_{\mu\nu}$ . The Weyl connection one-forms  $\Gamma^\mu_\nu$  are uniquely defined by the relations

$$d\omega^\mu + \Gamma^\mu_\nu \wedge \omega^\nu = 0,$$

$$\Gamma_{(\mu\nu)} = -g_{\mu\nu} \phi, \quad \text{where} \quad \Gamma_{ij} = g_{jk} \Gamma^k_j.$$

The curvature tensor  $R^\mu_{\nu\rho\sigma}$ , the Ricci tensor  $\text{Ric}_{\mu\nu}$  and the Ricci scalar  $R$  of a Weyl connection are defined as follows

$$R^\mu_\nu = d\Gamma^\mu_\nu + \Gamma^\mu_\rho \wedge \Gamma^\rho_\nu = \frac{1}{2} R^\mu_{\nu\rho\sigma} \omega^\rho \wedge \omega^\sigma$$

$$\text{Ric}_{\mu\nu} = R^\rho_{\mu\rho\nu},$$

$$R = \text{Ric}_{\mu\nu} g^{\mu\nu}.$$

The Ricci scalar has the conformal weight  $-2$ , that is it transforms as  $R \rightarrow e^{-2\lambda} R$  when  $g \rightarrow e^{2\lambda} g$ . Apart from these objects there is the Maxwell two-form

$$F = d\phi,$$

which is proportional to the antisymmetric part of the Ricci tensor.

Einstein-Weyl structures are by definition those Weyl structures for which the symmetric trace-free part of the Ricci tensor vanishes

$$\text{Ric}_{(\mu\nu)} - \frac{1}{n}R \cdot g_{\mu\nu} = 0.$$

4.2.2. *Einstein-Weyl structures from ODEs.* One sees from the system (57) that the pair  $(\widehat{g}, \Omega_3)$ , where

$$\widehat{g} = 2\theta^1\theta^3 - (\theta^2)^2$$

is Lie transported along the fibres of  $P^p \rightarrow \mathcal{S}$  in the following way

$$L_{X_4}g = \mathbf{A}_1^p(\theta^1)^2, \quad L_{X_5}\widehat{g} = 0, \quad L_{X_6}\widehat{g} = 0, \quad L_{X_7}\widehat{g} = 2\widehat{g},$$

and

$$L_{X_3}\Omega_3 = \frac{1}{2}\mathbf{C}_1^p\theta^1, \\ L_{X_j}\Omega_3 = 0, \quad \text{for } j = 5, 6, 7.$$

Due to these properties  $(\widehat{g}, \Omega_3)$  descends along  $P^p \rightarrow \mathcal{S}$  to the Weyl structure  $(g, \phi)$  on the solution space  $\mathcal{S}$  on condition that

$$W = 0$$

and

$$(58) \quad \left(\frac{1}{3}\mathcal{D}F_q - \frac{2}{9}F_q^2 - F_p\right)F_{qq} + \frac{2}{3}F_qF_{qp} - 2F_{qy} + F_{pp} = 0.$$

These conditions are equivalent to E. Cartan's original conditions in [3]. The conformal metric of the Weyl structure  $(g, \phi)$  coincides with the conformal metric of the contact case and is represented by

$$g = 2\omega^1\tilde{\omega}^3 - (\omega^2)^2 = \\ = 2(dy - pdx)(dq - \frac{1}{3}F_qdp + Kdy + (\frac{1}{3}qF_q - pK - F)dx) - (dp - qdx)^2,$$

while the Weyl potential is given by

$$\phi = -(2K_q + \frac{1}{9}F_{qq}F_q + \frac{1}{3}F_{qp})(dy - pdx) + \frac{1}{3}F_{qq}(dp - qdx) + \frac{1}{3}F_qdx.$$

The Weyl connection for this geometry, lifted to  $CO(2, 1) \rightarrow P^p \rightarrow \mathcal{S}$ , now the bundle of orthonormal frames, reads

$$\Gamma = \begin{pmatrix} -\Omega_1 & -\theta^4 & 0 \\ -\Omega_2 & -\Omega_3 & -\theta^4 \\ 0 & -\Omega_2 & \Omega_1 - 2\Omega_3 \end{pmatrix}.$$

The curvature is as follows

$$(59) \quad (R^\mu{}_\nu) = \begin{pmatrix} R^1_1 - F & R^1_2 & 0 \\ R^2_1 & -F & R^1_2 \\ 0 & R^2_1 & -R^1_1 - F \end{pmatrix}$$

with

$$F = d\Omega_3 = 2\mathbf{B}_3^p\theta^1\wedge\theta^2 + (2\mathbf{B}_4^p - 2\mathbf{B}_1^p)\theta^1\wedge\theta^3 - 2\mathbf{B}_2^p\theta^2\wedge\theta^3, \\ R^1_1 = -\mathbf{B}_3^p\theta^1\wedge\theta^2 - \mathbf{B}_4^p\theta^1\wedge\theta^3 - \mathbf{B}_2^p\theta^2\wedge\theta^3, \\ R^1_2 = \mathbf{B}_1^p\theta^1\wedge\theta^2 + \mathbf{B}_2^p\theta^1\wedge\theta^3, \\ R^2_1 = -\mathbf{B}_3^p\theta^1\wedge\theta^3 + (\mathbf{B}_1^p - 2\mathbf{B}_4^p)\theta^2\wedge\theta^3.$$

The Ricci tensor reads

$$\text{Ric} = \begin{pmatrix} 0 & -3\mathbf{B}_3^p & 3\mathbf{B}_1^p - 5\mathbf{B}_4^p \\ 3\mathbf{B}_3^p & 2\mathbf{B}_4^p & 3\mathbf{B}_2^p \\ -3\mathbf{B}_1^p + \mathbf{B}_1^p & -3\mathbf{B}_2^p & 0 \end{pmatrix}$$

and satisfies the Einstein-Weyl equations

$$\text{Ric}_{(ij)} = \frac{1}{3}R \cdot g_{ij}$$

with the Ricci scalar  $R = 6\mathbf{B}_4^p$ . The components of the curvature in the orthogonal coframe given by  $u_1 = 1, u_2 = 0, u_3 = 1$  are as follows

$$\begin{aligned}\mathbf{B}_1^p &= \frac{1}{18}F_{qqq}F_q + \frac{1}{6}F_{qqp} + \frac{1}{36}F_{qq}^2, \\ \mathbf{B}_2^p &= \frac{1}{6}F_{qqq}, \\ \mathbf{B}_3^p &= \frac{1}{6}F_{qqy} - \frac{1}{3}F_{qq}K_q - \frac{1}{6}F_{qqq}K - \frac{1}{18}F_{qq}F_{qp} - \frac{1}{54}F_{qq}^2F_q - L_q, \\ \mathbf{B}_4^p &= K_{qq} + \frac{1}{9}F_{qqq}F_q + \frac{1}{3}F_{qqp} + \frac{1}{12}F_{qq}^2.\end{aligned}$$

**Example 4.4.** The simplest nontrivial equations possessing the Einstein-Weyl geometry are

$$\begin{aligned}F &= \frac{3q^2}{2p}, & F &= \frac{3q^2p}{p^2+1}, \\ F &= \mu \frac{(2qy-p^2)^{3/2}}{y^2}, & F &= q^{3/2}.\end{aligned}$$

**4.3. Geometry on first jet space.** In section 3.5 we described how certain ODEs modulo contact transformations generate contact projective geometry on  $J^1$ . The fact that point transformations form a subclass within contact transformations suggests that ODEs modulo point transformations define some refined version of contact projective geometry. Indeed, the only object that is preserved by point transformations but is not preserved by contact transformations is the projection  $J^1 \rightarrow xy$  plane, whose fibres are generated by  $\partial_p$ . This motivate us to propose the following

**Definition 4.5.** A point projective structure on  $J^1$  is a contact projective structure, such that integral curves of the field  $\partial_p$  are geodesics of the contact projective structure.

We immediately get

**Lemma 4.6.** *The field  $\partial_p$  is geodesic for the contact projective geometry generated by an ODE provided that the ODE satisfies*

$$F_{qqq} = 0.$$

*Proof.* In the notation of section 3.5.1 we have  $\partial_p = e_2$  and, from proposition 3.8,  $\Gamma_{22}^1 = 0$ . Thus  $\nabla_2 e_2 = \lambda e_2$  iff  $\Gamma_{22}^3 = 0$ , which is equivalent to  $F_{qqq} = 0$  by means of lemma 3.9.  $\square$

However, the condition  $F_{qqq} = 0$  is not sufficient for  $\widehat{\omega}^p$  to be a Cartan connection for the point projective structure and we show that there does not exist any simple way to construct a Cartan connection on  $P^p \rightarrow J^1$ . The algebra  $\mathfrak{co}(2,1) \oplus \mathbb{R}^3 \subset \mathfrak{o}(3,2)$  inherits the following grading from (48)

$$\mathfrak{co}(2,1) \oplus \mathbb{R}^3 = \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1$$

but it is not semisimple, so the Tanaka method cannot be implemented. Moreover, the broadest generalization of this method – the Morimoto nilpotent geometry handling non-semisimple groups also fails in this case. This is because the Morimoto approach requires the algebra  $\mathfrak{g}$  to be equal to the prolongation of its non-positive part algebra  $\mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1} \oplus \mathfrak{g}_0$ . In our case the prolongation of  $\mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1} \oplus \mathfrak{g}_0$  is larger than  $\mathfrak{co}(2,1) \oplus \mathbb{R}^3$  and equals precisely  $\mathfrak{o}(3,2)$ , which yields a contact projective structure. Thus only the contact case is solved by the methods of the nilpotent geometry.

Lacking a general theory we must search a Cartan connection in a more direct way. Consider than an ODE satisfying  $F_{qqq} = 0$ . It follows that  $\mathbf{B}_2^p = 0$  and  $\mathbf{B}_4^p = \mathbf{B}_1^p$  in equations (57). We seek four one-forms

$$\begin{aligned}\Xi_1 &= \Omega_1 + a_1\theta^1 + a_2\theta^2 + a_3\theta^4, \\ \Xi_2 &= \Omega_2 + b_1\theta^1 + b_2\theta^2 + b_3\theta^4, \\ \Xi_3 &= \Omega_3 + c_1\theta^1 + c_2\theta^2 + c_3\theta^4, \\ \Xi_4 &= \theta^3 + f_1\theta^1 + f_2\theta^2 + f_3\theta^4,\end{aligned}$$

with yet unknown functions  $a_1, \dots, f_3$ , such that the matrix

$$\begin{pmatrix} \frac{1}{2}\Xi_1 & \frac{1}{2}\Xi_2 & 0 & 0 \\ \theta^4 & \Xi_3 - \frac{1}{2}\Xi_1 & 0 & 0 \\ \theta^2 & \Xi_4 & \frac{1}{2}\Xi_1 - \Xi_3 & -\frac{1}{2}\Xi_2 \\ 2\theta^1 & \theta^2 & -\theta^4 & -\frac{1}{2}\Xi_1 \end{pmatrix}$$

is a Cartan connection on  $P^p \rightarrow J^1$ . Calculating the curvature for this connection we obtain that the horizontality conditions yield

$$da_1 = X_1(a_1)\theta^1 + X_2(a_1)\theta^2 + \mathbf{B}_1^p\theta^3 + X_4(a_1)\theta^4 - a_1\Omega_1 - a_2\Omega_2.$$

Unfortunately, none combination of the structural functions  $\mathbf{A}_1^p, \dots, \mathbf{D}_2^p$  and their coframe derivatives of first order satisfies this condition. Therefore we are not able to build a Cartan connection for an arbitrary point projective structure. Moreover, since  $\mathbf{B}_1^p$  now equal to  $\frac{1}{a_3}(\frac{1}{36}F_{qq}^2 + \frac{1}{6}F_{qqp})$  is a basic relative invariant, it seems to us unlikely that among the coframe derivatives of  $\mathbf{A}_1^p, \dots, \mathbf{D}_2^p$  of any order there exists a function satisfying the above condition. If such a function existed it would mean that among the derivatives there is a more fundamental function from which  $\mathbf{B}_1^p$  can be obtained by differentiation.

Of course, we do have a Cartan connection for the point projective geometry provided that in addition to  $F_{qqq} = 0$  the conditions  $\mathbf{B}_1^p = \mathbf{D}_1^p = 0$  are imposed. However, the geometric interpretation of these conditions is unclear.

**4.4. Six-dimensional Weyl geometry in the split signature.** The construction of the six-dimensional split signature conformal geometry given in section 3 has also its Weyl counterpart in the point case. A similar construction was done by P. Nurowski, [38] but he considered the conformal metric, not the Weyl geometry. Here, apart from the tensor

$$\widehat{\mathbf{g}} = 2(\Omega_1 - \Omega_3)\theta^2 - 2\Omega_3\theta^1 + 2\theta^4\theta^3$$

of (49), we also have the one-form

$$\frac{1}{2}\Omega_3.$$

The Lie derivatives along the degenerate direction  $X_5 + X_7$  of  $\widehat{\mathbf{g}}$  are

$$L_{(X_5+X_7)}\widehat{\mathbf{g}} = \widehat{\mathbf{g}} \quad \text{and} \quad L_{(X_5+X_7)}\Omega_3 = 0.$$

In this manner the pair  $(\widehat{\mathbf{g}}, \frac{1}{2}\Omega_3)$  generates the six-dimensional split-signature Weyl geometry  $(\mathbf{g}, \phi)$  on the six-manifold  $M^6$  being the space of integral curves of  $X_5 + X_7$ .

The associated Weyl connection is  $\mathfrak{co}(3,3) \oplus \mathbb{R}^6$ -valued and has the following form.

$$\Gamma^\mu_\nu = \begin{pmatrix} 0 & \frac{1}{2}\theta^4 & \Gamma^1_3 & 0 & \Gamma^1_5 & \Gamma^1_6 \\ \frac{1}{2}\Omega_2 & \frac{1}{2}\Omega_1 - \frac{1}{2}\Omega_3 & \Gamma^2_3 & -\Gamma^1_5 & 0 & \Gamma^2_6 \\ \frac{1}{2}\theta^4 & 0 & \frac{1}{2}\theta^3 - \frac{1}{2}\theta^1 & -\Gamma^1_6 & -\Gamma^2_6 & 0 \\ 0 & -\frac{1}{2}\theta^1 & \frac{1}{2}\theta^3 & -\Omega_3 & -\frac{1}{2}\Omega_2 & -\frac{1}{2}\theta^4 \\ \frac{1}{2}\theta^1 & 0 & \frac{1}{2}\theta^2 & -\frac{1}{2}\theta^4 & -\frac{1}{2}\Omega_1 - \frac{1}{2}\Omega_3 & 0 \\ -\frac{1}{2}\theta^3 & -\frac{1}{2}\theta^2 & 0 & -\Gamma^1_3 & -\Gamma^2_3 & \frac{1}{2}\Omega_1 - \frac{3}{2}\Omega_3 \end{pmatrix},$$

where

$$\begin{aligned} \Gamma^1_3 &= \frac{1}{2}\Omega_2 + \frac{1}{2}\mathbf{A}_2^p\theta^1, \\ \Gamma^2_3 &= \mathbf{A}_3^p\theta^1 + \frac{1}{2}\mathbf{A}_2^p\theta^2 + \mathbf{A}_1^p\theta^4, \\ \Gamma^1_5 &= \mathbf{D}_1^p\theta^1 + \mathbf{B}_3^p\theta^2 + (\frac{3}{2}\mathbf{B}_4^p - \mathbf{B}_1^p)\theta^3 + (\mathbf{C}_1^p - \frac{1}{2}\mathbf{A}_2^p)\theta^4, \\ \Gamma^1_6 &= (\frac{1}{2}\mathbf{B}_4^p - \mathbf{B}_1^p)\theta^1 - \mathbf{B}_2^p\theta^2, \\ \Gamma^2_6 &= (\mathbf{B}_3^p + \mathbf{D}_1^p)\theta^1 + \frac{1}{2}\mathbf{B}_4^p\theta^2 + \mathbf{B}_2^p\theta^4. \end{aligned}$$

Contrary to section 3.6, this connection seems to have full holonomy  $CO(3,3) \ltimes \mathbb{R}^6$  and it is not generated by  $\widehat{\omega}^p$  in any simple way. It is also never Einstein-Weyl.

**4.5. Three-dimensional Lorentzian geometry on solution space.** The geometries of sections 4.2 to 4.4 are counterparts of respective geometries of the contact case. The point classification, however, contains another geometry, which is new when compared to the contact case. This is owing to the fact that the Einstein-Weyl geometry of section 4.2 has in general the non-vanishing Ricci scalar, which is a weighted function and can be fixed to a constant by an appropriate choice of the conformal gauge. Thereby the Weyl geometry on  $\mathcal{S}$  is reduced to a Lorentzian metric geometry.

These properties of the Weyl geometry are reflected at the level of the ODEs by the fact that the equations

$$(60) \quad y''' = \frac{3}{2} \frac{(y'')^2}{y'}$$

and

$$(61) \quad y''' = \frac{3y'(y'')^2}{y'^2 + 1}$$

are contact equivalent to the trivial  $y''' = 0$  by means of corollary 3.3 but they are mutually *non-equivalent* under point transformations and possess the  $\mathfrak{o}(2,2)$  and  $\mathfrak{o}(4)$  algebra of point symmetries respectively. Both of them generate the same flat conformal geometry but their Weyl geometries differ. After calculating equations (59) we see that the only non-vanishing component of their curvature is the Ricci scalar, which is negative for the equation (60) and positive for (61). In this circumstances we do another reduction step in the Cartan algorithm setting the Ricci scalar equal to  $\pm 6$  respectively<sup>6</sup>, which means  $\mathbf{B}_4^p = \pm 1$ , and obtain a six-dimensional subbundle  $P_6^p$  of  $P^p$ . The invariant coframe  $(\theta^1, \theta^2, \theta^3, \theta^4, \Omega_1, \Omega_2)$  yields

<sup>6</sup>We choose  $\pm 6$  here to avoid large numerical factors.

the local structure of  $SO(2, 2)$  or  $SO(4)$  on  $P_6^p$  while the tensor  $\widehat{g} = 2\theta^1\theta^3 - (\theta^2)^2$  descends to a metric rather than a conformal class on  $\mathcal{S}$  by means of conditions

$$(62) \quad L_{X_5}\widehat{g} = 0, \quad L_{X_6}\widehat{g} = 0.$$

The obtained metrics are locally diffeomorphic to the metrics on the symmetric spaces  $SO(2, 2)/SO(2, 1)$  or  $SO(4)/SO(3)$ .

In order to generalize this construction to a broader class of equations we assume that the Ricci scalar of the Einstein-Weyl geometry is non-zero

$$6K_{qq} + \frac{2}{3}F_{qqq}F_q + 2F_{qqp} + \frac{1}{2}F_{qq}^2 \neq 0$$

and set

$$u_3 = \sqrt{\left| 6K_{qq} + \frac{2}{3}F_{qqq}F_q + 2F_{qqp} + \frac{1}{2}F_{qq}^2 \right|}$$

in the coframe of theorem 4.1. The tensor  $\widehat{g}$  on  $P_6^p$  projects to the metric  $g$  on  $\mathcal{S}$  provided that the conditions (62) still hold, which is equivalent to

$$W = 0 \quad \text{and} \quad (\mathcal{D} + \frac{2}{3}F_q) \left( 6K_{qq} + \frac{2}{3}F_{qqq}F_q + 2F_{qqp} + \frac{1}{2}F_{qq}^2 \right) = 0.$$

The Cartan coframe on  $P_6^p$  is then given by

$$\begin{aligned} d\theta^1 &= \Omega_1 \wedge \theta^1 - \theta^2 \wedge \theta^4, \\ d\theta^2 &= \Omega_2 \wedge \theta^1 + \mathbf{p}_1 \theta^2 \wedge \theta^3 - \theta^3 \wedge \theta^4, \\ d\theta^3 &= \Omega_2 \wedge \theta^2 - \Omega_1 \wedge \theta^3 + \mathbf{p}_2 \theta^2 \wedge \theta^3, \\ d\theta^4 &= \Omega_1 \wedge \theta^4 + \mathbf{p}_3 \theta^1 \wedge \theta^2 + \mathbf{p}_4 \theta^1 \wedge \theta^3 + \mathbf{p}_5 \theta^1 \wedge \theta^4 - \frac{1}{2} \mathbf{p}_2 \theta^2 \wedge \theta^4 + \mathbf{p}_1 \theta^3 \wedge \theta^4, \\ d\Omega_1 &= -\Omega_2 \wedge \theta^4 + \mathbf{p}_2 \Omega_2 \wedge \theta^1 + \mathbf{p}_6 \theta^1 \wedge \theta^2 + \mathbf{p}_7 \theta^1 \wedge \theta^3 + \mathbf{p}_4 \theta^2 \wedge \theta^3 + \mathbf{p}_5 \theta^2 \wedge \theta^4, \\ d\Omega_2 &= -\Omega_1 \wedge \Omega_2 + \mathbf{p}_1 \Omega_2 \wedge \theta^3 + \mathbf{p}_8 \theta^1 \wedge \theta^2 + \mathbf{p}_9 \theta^1 \wedge \theta^3 + \mathbf{p}_{10} \theta^2 \wedge \theta^3 + \mathbf{p}_5 \theta^3 \wedge \theta^4, \end{aligned}$$

with some functions  $\mathbf{p}_1, \dots, \mathbf{p}_{10}$  and the Levi-Civita connection is given by

$$\begin{pmatrix} \Gamma^1_1 & \Gamma^1_2 & 0 \\ \Gamma^2_1 & 0 & \Gamma^1_2 \\ 0 & \Gamma^2_1 & -\Gamma^1_1 \end{pmatrix},$$

where

$$\begin{aligned} \Gamma^1_1 &= -\Omega_1 + \frac{1}{2} \mathbf{p}_2 \theta^2, \\ \Gamma^1_2 &= \frac{1}{2} \mathbf{p}_2 \theta^1 - \mathbf{p}_1 \theta^2 - \theta^4, \\ \Gamma^2_1 &= -\Omega_2 + \frac{1}{2} \mathbf{p}_2 \theta^3. \end{aligned}$$

The curvature reads

$$\begin{pmatrix} R^1_1 & R^1_2 & 0 \\ R^2_1 & 0 & R^1_2 \\ 0 & R^2_1 & -R^1_1 \end{pmatrix},$$

$$R^1_1 = \frac{1}{2}(\mathbf{p}_9 - \mathbf{p}_6)\theta^1 \wedge \theta^2 + (\frac{1}{4}(\mathbf{p}_2)^2 - \mathbf{p}_7)\theta^1 \wedge \theta^3 + (\mathbf{p}_4 + X_2(\mathbf{p}_1) + \frac{1}{2}\mathbf{p}_1\mathbf{p}_2)\theta^2 \wedge \theta^3,$$

$$\begin{aligned} R^1_2 &= (\mathbf{p}_{10} - \frac{1}{2}X_2(\mathbf{p}_2) - \frac{1}{4}(\mathbf{p}_2)^2)\theta^1 \wedge \theta^2 + (\mathbf{p}_4 + X_2(\mathbf{p}_1) + \frac{1}{2}\mathbf{p}_1\mathbf{p}_2)\theta^1 \wedge \theta^3 \\ &\quad + ((\mathbf{p}_1)^2 - X_3(\mathbf{p}_1))\theta^2 \wedge \theta^3, \end{aligned}$$

$$R^2_1 = -\mathbf{p}_8\theta^1 \wedge \theta^2 + \frac{1}{2}(\mathbf{p}_6 - \mathbf{p}_9)\theta^1 \wedge \theta^3 + (-\mathbf{p}_{10} + \frac{1}{2}X_2(\mathbf{p}_2) + \frac{1}{4}(\mathbf{p}_2)^2)\theta^2 \wedge \theta^3.$$

## 5. GEOMETRY OF THIRD-ORDER ODES MODULO FIBRE-PRESERVING TRANSFORMATIONS OF VARIABLES

**5.1. Cartan connection on seven-dimensional bundle.** The construction of a Cartan connection for the fibre preserving case is very similar to its point counterpart. This is due to the fact that every point symmetry of  $y''' = 0$  is necessarily fibre-preserving and, as a consequence, the bundle we will construct is also of dimension seven. Starting from the  $G_f$ -structure of (14), which is given by the forms

$$\begin{aligned}\theta^1 &= u_1 \omega^1, \\ \theta^2 &= u_2 \omega^1 + u_3 \omega^2, \\ \theta^3 &= u_4 \omega^1 + u_5 \omega^2 + u_6 \omega^3, \\ \theta^4 &= u_7 \omega^4,\end{aligned}$$

and after the substitutions

$$\begin{aligned}u_6 &= \frac{u_3^2}{u_1}, & u_7 &= \frac{u_1}{u_3}, \\ u_5 &= \frac{u_3}{u_1} \left( u_2 - \frac{1}{3} u_3 F_q \right), \\ u_4 &= \frac{u_3^2}{u_1} K + \frac{u_2^2}{2u_1}\end{aligned}$$

we get the following theorem.

**Theorem 5.1.** *To every third order ODE  $y''' = F(x, y, y', y'')$  there are associated the following data.*

- i) *The principal fibre bundle  $H_3 \rightarrow P^f \rightarrow J^2$ , where  $\dim P^f = 7$ , and  $H_3$  is the three-dimensional group the same as in the point case*

$$H_3 = \begin{pmatrix} \sqrt{u_1}, & \frac{1}{2} \frac{u_2}{\sqrt{u_1}}, & 0 & 0 \\ 0 & \frac{u_3}{\sqrt{u_1}}, & 0 & 0 \\ 0 & 0 & \frac{\sqrt{u_1}}{u_3}, & -\frac{1}{2} \frac{u_2}{\sqrt{u_1} u_3} \\ 0 & 0 & 0 & \frac{1}{\sqrt{u_1}} \end{pmatrix},$$

- ii) *The coframe  $(\theta^1, \theta^2, \theta^3, \theta^4, \Omega_1, \Omega_2, \Omega_3)$ , which defines the  $\mathfrak{co}(2, 1) \oplus \mathbb{R}^3$ -valued Cartan connection  $\hat{\omega}^f$  on  $P^f$  by*

$$(63) \quad \hat{\omega}^f = \begin{pmatrix} \frac{1}{2} \Omega_1 & \frac{1}{2} \Omega_2 & 0 & 0 \\ \theta^4 & \Omega_3 - \frac{1}{2} \Omega_1 & 0 & 0 \\ \theta^2 & \theta^3 & \frac{1}{2} \Omega_1 - \Omega_3 & -\frac{1}{2} \Omega_2 \\ 2\theta^1 & \theta^2 & -\theta^4 & -\frac{1}{2} \Omega_1 \end{pmatrix}$$

Two 3rd order ODEs  $y''' = F(x, y, y', y'')$  and  $y''' = \bar{F}(x, y, y', y'')$  are locally fibre-preserving equivalent if and only if their associated Cartan connections are locally diffeomorphic, that is there exists a local bundle diffeomorphism  $\Phi: \bar{P}^f \supset \bar{\mathcal{O}} \rightarrow \mathcal{O} \subset P^f$  such that  $\Phi^* \hat{\omega}^f = \bar{\omega}^f$ . The value of  $\hat{\omega}^f$  at the point  $(x^i, u_\mu)$  in  $P^f$  is given by

$$\hat{\omega}^f(x^i, u_\mu) = u^{-1} \omega^f u + u^{-1} du$$



where  $u \in H_3$  and

$$\omega^f = \begin{pmatrix} \frac{1}{2}\Omega_1^0 & \frac{1}{2}\Omega_2^0 & 0 & 0 \\ \tilde{\omega}^4 & \Omega_3^0 - \frac{1}{2}\Omega_1^0 & 0 & 0 \\ \omega^2 & \tilde{\omega}^3 & \frac{1}{2}\Omega_1^0 - \Omega_3^0 & -\frac{1}{2}\Omega_2^0 \\ 2\omega^1 & \omega^2 & -\tilde{\omega}^4 & -\frac{1}{2}\Omega_1^0 \end{pmatrix}$$

is given by

$$\begin{aligned} \omega^1 &= dy - p dx, \\ \omega^2 &= dp - q dx, \\ \tilde{\omega}^3 &= dq - F dx - \frac{1}{3}F_q(dp - q dx) + K(dy - p dx), \\ \omega^4 &= dx \\ \Omega_1^0 &= -K_q \omega^1 + \frac{1}{3}F_{qq} \omega^2, \\ \Omega_2^0 &= L \omega^1 - K_q \omega^2 + \frac{1}{3}F_{qq} \tilde{\omega}^3 - K \omega^4, \\ \Omega_3^0 &= -K_q \omega^1 + \frac{1}{3}F_{qq} \omega^2 + \frac{1}{3}F_q \omega^4. \end{aligned}$$

The exterior differentials of the coframe are equal to

$$\begin{aligned} d\theta^1 &= \Omega_1 \wedge \theta^1 + \theta^4 \wedge \theta^2 + \mathbf{B}_1^f \theta^1 \wedge \theta^2, \\ d\theta^2 &= \Omega_2 \wedge \theta^1 + \Omega_3 \wedge \theta^2 + \theta^4 \wedge \theta^3 + \mathbf{B}_1^f \theta^1 \wedge \theta^3, \\ d\theta^3 &= \Omega_2 \wedge \theta^2 + (2\Omega_3 - \Omega_1) \wedge \theta^3 + \mathbf{A}_1^f \theta^4 \wedge \theta^1 + \mathbf{B}_1^f \theta^2 \wedge \theta^3, \\ d\theta^4 &= (\Omega_1 - \Omega_3) \wedge \theta^4, \\ (64) \quad d\Omega_1 &= -\Omega_2 \wedge \theta^4 + (\mathbf{D}_1^f - \mathbf{B}_2^f) \theta^1 \wedge \theta^2 + \mathbf{B}_3^f \theta^1 \wedge \theta^3 + \\ &\quad + (2\mathbf{C}_1^f - \mathbf{A}_2^f) \theta^1 \wedge \theta^4 + \mathbf{B}_4^f \theta^2 \wedge \theta^3 + \mathbf{B}_5^f \theta^2 \wedge \theta^4, \\ d\Omega_2 &= (\Omega_3 - \Omega_1) \wedge \Omega_2 + \mathbf{D}_2^f \theta^1 \wedge \theta^2 + (\mathbf{D}_1^f - 2\mathbf{B}_2^f) \theta^1 \wedge \theta^3 + \mathbf{A}_3^f \theta^1 \wedge \theta^4 \\ &\quad + \mathbf{B}_6^f \theta^2 \wedge \theta^3 + (\mathbf{C}_1^f - \mathbf{A}_2^f) \theta^2 \wedge \theta^4 + \mathbf{B}_5^f \theta^3 \wedge \theta^4, \\ d\Omega_3 &= (\mathbf{D}_1^f - \mathbf{B}_2^f) \theta^1 \wedge \theta^2 + \mathbf{B}_3^f \theta^1 \wedge \theta^3 + (\mathbf{C}_1^f - \mathbf{A}_2^f) \theta^1 \wedge \theta^4 \\ &\quad + \mathbf{B}_4^f \theta^2 \wedge \theta^3 + \frac{1}{2}\mathbf{B}_5^f \theta^2 \wedge \theta^4, \end{aligned}$$

where  $\mathbf{A}_1^f, \mathbf{A}_2^f, \mathbf{A}_3^f, \mathbf{B}_1^f, \mathbf{B}_2^f, \mathbf{B}_3^f, \mathbf{B}_4^f, \mathbf{B}_5^f, \mathbf{B}_6^f, \mathbf{C}_1^f, \mathbf{D}_1^f, \mathbf{D}_2^f$  are functions on  $P^f$ . All these invariants express by the coframe derivatives of  $\mathbf{A}_1^f, \mathbf{B}_1^f, \mathbf{C}_1^f$ , which read

$$\begin{aligned} \mathbf{A}_1^f &= \frac{u_3^3}{u_1^3} W, \\ \mathbf{B}_1^f &= \frac{1}{3u_3} F_{qq}, \\ \mathbf{C}_1^f &= \frac{u_2}{u_1^2} \left( \frac{1}{9} F_{qq} F_q + \frac{1}{3} F_{qp} + K_q \right) + \\ &\quad + \frac{u_3}{u_1^2} \left( \frac{2}{3} F_{qq} K - \frac{1}{3} K_q F_q - K_p - \frac{2}{3} F_{qy} \right). \end{aligned}$$

If  $\mathbf{A}_1^f = 0$  then  $\mathbf{A}_2^f = \mathbf{A}_3^f = 0$  and if  $\mathbf{B}_1^f = 0$  then  $\mathbf{B}_i^f = 0$  for  $i = 2, \dots, 6$ . In particular we have

$$d\mathbf{B}_1^f = -\mathbf{B}_2^f \theta^1 + (\mathbf{B}_6^f - \mathbf{B}_3^f) \theta^2 - \mathbf{B}_4^f \theta^3 - \mathbf{B}_5^f \theta^4 - \mathbf{B}_1^f \Omega_3.$$

The flat case is given by vanishing of  $\mathbf{A}_1^f, \mathbf{B}_1^f$  and  $\mathbf{C}_1^f$ .

## 5.2. Fibre-preserving geometry from point geometry.

5.2.1. *Fibre-preserving versus point objects.* An immediate observation about the fibre-preserving objects –  $P^f$  and  $\widehat{\omega}^f$  – is that they are closely related to their point counterparts of theorem 4.1. Indeed, there is a unique diffeomorphism  $\rho: P^p \rightarrow P^f$  such that

$$(65) \quad \theta^p_1 = \rho^* \theta^f_1, \quad \theta^p_2 = \rho^* \theta^f_2, \quad \theta^p_3 = \rho^* \theta^f_3.$$

It is given by the identity map in the coordinate systems of theorems 4.1 and 5.1. The remaining one-forms are transported as follows.

$$(66) \quad \begin{aligned} \theta^p_4 &= \rho^* (\theta^f_4 + \tfrac{1}{2} \mathbf{B}_1^f \theta^f_1), \\ \Omega_1^p &= \rho^* (\Omega_1^f + \mathbf{B}_5^f \theta^f_1 - \tfrac{1}{2} \mathbf{B}_1^f \theta^f_2), \\ \Omega_2^p &= \rho^* (\Omega_2^f + \tfrac{1}{2} \mathbf{B}_5^f \theta^f_2 - \tfrac{1}{2} \mathbf{B}_1^f \theta^f_3), \\ \Omega_3^p &= \rho^* (\Omega_3^f + \tfrac{1}{2} \mathbf{B}_5^f \theta^f_1), \end{aligned}$$

where

$$\mathbf{B}_5^f = -\frac{1}{3u_1} (\mathcal{D}(F_{qq}) + \tfrac{1}{3} F_{qq} F_q).$$

The above formulae enable us to transform easily the fibre-preserving coframe into the point coframe. Given the fibre-preserving coframe  $(\theta^f_1, \dots, \Omega_3^f)$  we compute  $d\theta^f_1$  and take the coefficient of the  $\theta^f_1 \wedge \theta^f_2$  term. This is the function  $\mathbf{B}_1^f$ . Next we compute  $d\mathbf{B}_1^f$ , decompose it in the fibre-preserving coframe, take minus function that stands next to  $\theta^f_4$  and this is  $\mathbf{B}_5^f$ . We substitute these functions together with  $(\theta^f_1, \dots, \Omega_3^f)$  into the right hand side of (65) and (66), where  $\rho$  is the identity transformation of  $P^f$  and the point coframe is explicitly constructed on  $P^f$ .

Now let us consider the inverse construction, from the fibre-preserving case to the point case. If we have only the point coframe  $(\theta^p_1, \dots, \Omega_3^p)$  then we can not utilize eq. (66) since we are not able to construct the function  $\mathbf{B}_1^f$ , which is not a point invariant<sup>7</sup> and, as such, does not appear among functions  $\mathbf{A}_1^p, \dots, \mathbf{D}_2^p$  in (57) or among their derivatives. However, if we consider the point coframe *and* the function  $\mathbf{B}_1^f$  then the construction is possible, since  $\mathbf{B}_5^f$  is given by the derivative  $-X_4(\mathbf{B}_1^f)$  along the field  $X_4$  of the *point* dual frame. Therefore the passage from the point case to the fibre-preserving case is possible if we supplement the connection  $\widehat{\omega}^p$  with the function  $\mathbf{B}_1^f$ . This fact implies that each construction of the point case has its fibre-preserving counterpart which has an additional object generated by  $\mathbf{B}_1^f$ .

5.2.2. *Counterpart of the Einstein-Weyl geometry on  $\mathcal{S}$ .* This geometry is constructed in the following way. Let  $(\theta^1, \dots, \Omega_3)$  denotes again the fibre-preserving coframe. Given the objects  $\widehat{g} = 2\theta^1\theta^3 - (\theta^2)^2$  and

$$\widehat{\phi} = \Omega_3 + \tfrac{1}{2} \mathbf{B}_5^f \theta^1,$$

let us also consider the function  $\mathbf{B}_1^f$ , and ask under what conditions the triple  $(\widehat{g}, \widehat{\phi}, \mathbf{B}_1^f)$  can be projected to a geometry on  $\mathcal{S}$ . There are two possibilities here,

<sup>7</sup>For example the point transformation  $(x, y) \rightarrow (y, x)$  destroys the condition  $F_{qq} = 0$ .

either  $\mathbf{B}_1^f = 0$  or  $\mathbf{B}_1^f \neq 0$ . If  $\mathbf{B}_1^f = 0$  then it is easy to see that the pair  $(\widehat{g}, \widehat{\phi})$  generates the Einstein-Weyl geometry if only  $\mathbf{A}_1^f = \mathbf{C}_1^f = 0$ , which means that we are in the trivial case  $y''' = 0$ .

Suppose  $\mathbf{B}_1^f \neq 0$  then. For the geometry on  $\mathcal{S}$  to exist we need not only the conditions for the Lie transport of  $\widehat{g}$  and  $\widehat{\phi}$  but also

$$(67) \quad L_{X_i} \mathbf{B}_1^f = 0, \quad \text{for } i = 4, 5, 6, \quad L_{X_7} \mathbf{B}_1^f = -\mathbf{B}_1^f.$$

If all these conditions are satisfied then  $(\widehat{g}, \widehat{\phi}, \mathbf{B}_1^f)$  defines on  $\mathcal{S}$  the Einstein-Weyl geometry  $(g, \phi)$  of the point case, which is equipped with an additional object: a weighted function  $f$  which transforms  $f \rightarrow e^{-\lambda} f$  when  $g \rightarrow e^{2\lambda} g$  and is given by the projection of  $\mathbf{B}_1^f$ . The conditions for existence of this geometry are  $\mathbf{A}_1^f = \mathbf{B}_5^f = 0$ , that is

$$(68) \quad W = 0 \quad \text{and} \quad \mathcal{D}(F_{qq}) + \frac{1}{3} F_{qq} F_q = 0.$$

As usual, the condition  $W = 0$  guarantees existence of  $[g]$  and the other condition yields (67). The proper Lie transport of  $\widehat{\phi}$  along  $X_4$  is already guaranteed by the above conditions as their differential consequence.

5.2.3. *Counterpart of the Weyl geometry on  $M^6$ .* In the similar vein we show that the triple  $(\widehat{\mathbf{g}}, \frac{1}{2}\widehat{\phi}, \mathbf{B}_1^f)$ , where

$$\begin{aligned} \widehat{\mathbf{g}} &= 2(\Omega_1 - \Omega_3)\theta^2 - 2\Omega_3\theta^1 + 2\theta^4\theta^3, \\ \widehat{\phi} &= \Omega_3 + \frac{1}{2}\mathbf{B}_5^f\theta^1. \end{aligned}$$

projects to the six-dimensional split signature Weyl geometry  $(\mathbf{g}, \phi)$  of section 4 section 4.4 equipped with a function  $f$  of conformal weight  $-2$ .

5.2.4. *Counterpart of Lorentzian geometry on  $\mathcal{S}$ .* Given  $(g, \phi, f)$  on  $\mathcal{S}$  it is natural to fix the conformal gauge so as<sup>8</sup>  $f = 1$ . This is equivalent to another substitution

$$u_3 = \frac{1}{3} F_{qq}$$

in the Cartan reduction algorithm, which leads us to the bundle  $P_6^f$  with the following differential system

$$(69) \quad \begin{aligned} d\theta^1 &= \Omega_1 \wedge \theta^1 + \theta^4 \wedge \theta^2, \\ d\theta^2 &= \Omega_2 \wedge \theta^1 + \mathbf{f}_1 \theta^3 \wedge \theta^2 + \mathbf{f}_2 \theta^4 \wedge \theta^2 + \theta^4 \wedge \theta^3, \\ d\theta^3 &= -\Omega_1 \wedge \theta^3 + \Omega_2 \wedge \theta^2 + (2 - 2\mathbf{f}_3) \theta^3 \wedge \theta^2 + \mathbf{f}_4 \theta^4 \wedge \theta^1 + 2\mathbf{f}_2 \theta^4 \wedge \theta^3, \\ d\theta^4 &= \Omega_1 \wedge \theta^4 + \mathbf{f}_5 \theta^4 \wedge \theta^1 + (\mathbf{f}_3 - 2) \theta^4 \wedge \theta^2 + \mathbf{f}_1 \theta^4 \wedge \theta^3, \\ d\Omega_1 &= (2\mathbf{f}_3 - 2) \Omega_2 \wedge \theta^1 - \Omega_2 \wedge \theta^4 + \mathbf{f}_6 \theta^1 \wedge \theta^2 + \mathbf{f}_7 \theta^1 \wedge \theta^3 + \mathbf{f}_8 \theta^1 \wedge \theta^4 - \mathbf{f}_5 \theta^2 \theta^4, \\ d\Omega_2 &= \Omega_2 \wedge \Omega_1 - \mathbf{f}_1 \Omega_2 \wedge \theta^3 - \mathbf{f}_2 \Omega_2 \wedge \theta^4 + \mathbf{f}_9 \theta^1 \wedge \theta^2 + \mathbf{f}_{10} \theta^1 \wedge \theta^3 + \mathbf{f}_{11} \theta^1 \wedge \theta^4 + \\ &\quad + \mathbf{f}_{12} \theta^2 \wedge \theta^3 + \mathbf{f}_{13} \theta^2 \wedge \theta^4 - \mathbf{f}_5 \theta^3 \wedge \theta^4. \end{aligned}$$

If the conditions (68), now equivalent to  $\mathbf{f}_4 = 0 = \mathbf{f}_2$ , are satisfied then  $\widehat{g} = 2\theta^1\theta^3 - (\theta^2)^2$  projects to a Lorentzian metric  $g$  and

$$\widehat{\phi} = -2\mathbf{f}_5\theta^1 + 2\mathbf{f}_3\theta^2 + 2\mathbf{f}_1\theta^3$$

projects to a one-form  $\phi$ . With the Lorentzian metric there is associated the Levi-Civita connection  $(\Gamma^\mu_\nu)$ :

$$\begin{aligned} \Gamma^1_1 &= -\Omega_1 + (\mathbf{f}_3 - 1)\theta^2, \\ \Gamma^1_2 &= (\mathbf{f}_3 - 1)\theta^1 + \mathbf{f}_1\theta^2 - \theta^4, \\ \Gamma^2_1 &= -\Omega_2 + (\mathbf{f}_3 - 1)\theta^3. \end{aligned}$$

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<sup>8</sup>With possible change of the signature to make  $f$  positive.

The covariant derivative of  $\phi$  with respect to  $\Gamma^\mu_\nu$  is as follows

$$\phi_{i;j} = \begin{pmatrix} -\mathbf{f}_9 - (\mathbf{f}_5)^2 & \frac{1}{2}\mathbf{f}_6 + \mathbf{f}_5(\mathbf{f}_3 - 2) & \mathbf{f}_{12} - \mathbf{f}_3(\mathbf{f}_3 - 3) \\ \frac{1}{2}\mathbf{f}_6 + \mathbf{f}_5\mathbf{f}_3 & 2\mathbf{f}_{12} - 2\mathbf{f}_3(\mathbf{f}_3 - 2) & X_2(\mathbf{f}_1) + \mathbf{f}_1 \\ \mathbf{f}_{12} - \mathbf{f}_3(\mathbf{f}_3 - 1) & X_2(\mathbf{f}_1) - \mathbf{f}_1 & X_3(\mathbf{f}_1) \end{pmatrix}.$$

The one-form  $\phi$  and the Ricci tensor satisfy the following identities

$$\begin{aligned} \nabla_{(i}\phi_{j)} &= -\text{Ric}_{ij} - \phi_i\phi_j + (\phi^k\phi_k + 2)g_{ij}, \\ \text{R} &= 2\phi^k\phi_k + 6, \\ d\phi &= -2 * \phi. \end{aligned}$$

The homogeneous model of this geometry is associated to  $y''' = \frac{3}{2} \frac{(y'')^2}{y'}$ .

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